Advanced Quantum Field Theory

Exercise 9

Consider the system of two real massive scalar fields with cubic interaction:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2 + \frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}M^2\Phi^2 - \frac{k}{2}\varphi^2\Phi \tag{1}$$

Take the field Φ much heavier than the field φ, $M \gg m$, k, and consider the effective Lagrangian describing only the light degree of freedom:

$$\mathcal{L}_{eff} = \frac{1}{2}\left(1 + C_1^{(2,2)} + \dots\right)\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}\left(m^2 + C_1^{(2,0)} + \dots\right)\varphi^2 - \frac{1}{4!}\left(C_0^{(4,0)} + C_1^{(4,0)} + \dots\right)\varphi^4\Phi + \frac{1}{3!}\left(C_0^{(4,2)} + \dots\right)\varphi^3 \tag{2}$$

In the above expression the coefficients $C_l^{(a,b)}$ are the coefficients associated to the operators with (# of fields,# of derivatives)=(a,b) matched with the full theory at l-loops. Notice in addition that we have included only one operator containing 4 fields and 2 derivatives. Indeed, modulo total derivatives, this is the only independent structure that can arise (prove it).

Show that the effects of integrating out the heavy field can be obtained in one shot using the classical equation of motion. In order to prove this result compute the Euler-Lagrangian equation for Φ, solve it formally for Φ and plug this expression into the full Lagrangian. In order to simplify the computation notice that the part of the Lagrangian containing Φ can be written as:

$$\mathcal{L} \supset -\frac{1}{2}\Phi\underbrace{(\text{eq. of motion of }\Phi)}_{=0} - \frac{k}{4}\Phi\varphi^2 + \dots \tag{3}$$

Expand formally the found expression in $\Box/M^2$ and extract the tree level coefficients.

Compute the coefficients $C_1^{(2,2)}$, $C_1^{(4,0)}$, $C_0^{(4,2)}$ matching the effective field theory with the complete theory.