Quantum Field Theory

Set 8

Exercise 1: Noether's currents in classical mechanics

Consider the general Lagrangian of a classical system

$$L = \sum_a \frac{m_a}{2} (\dot{\vec{q}}_a)^2 - \sum_{a \neq b} V\left(|\vec{q}_a - \vec{q}_b|\right) .$$

Using the Noether procedure compute the Noether current associated to the following transformations

- time translations;
- coordinate translations;
- coordinate rotations.

Exercise 2: Noether's current for $U(1)$ global symmetry

Consider the Lagrangian of a free *complex* massive scalar field $\phi(x) = \phi_1(x) + i\phi_2(x)$:

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi \, .$$

Show that the Lagrangian density is invariant under the following transformation

$$x_\mu \longrightarrow x_\mu \, , \qquad \phi(x) \longrightarrow e^{i\alpha} \phi(x),$$

where α does not depend on x.
Compute the Noether current associated to this symmetry and show that is conserved only if one uses the equations of motion.

Exercise 3: Casimirs of Poincaré algebra

To build the irreducible unitary representation under which the physical states transform in a quantum field theory, one should find the Casimir operators of the Poincaré algebra, i.e. all the operators which commute with its generators. For instance the operator P^2 is such a Casimir and its value determines the mass of the physical state.
Find the other Casimir of the Poincaré algebra. *Hint:* start by building an operator W^μ invariant under translations and transforming as a vector under Lorentz transformations. Does $W^\mu = P_\nu J^{\mu\nu}$ work?
Consider the representation of a massive particle in the subspace where the particle is at rest, $P^\mu = (M, 0, 0, 0)$. What does the vector W^μ correspond to?

Exercise 4: Noether's charge as generator of transformations

Given a Lagrangian density $\mathcal{L}(\phi_a, \partial_\mu \phi_b)$ consider the symmetry transformation defined by

$$\begin{aligned}
&x^\mu \longrightarrow x'^\mu = f(x) \simeq x^\mu - \epsilon^\mu_i \alpha^i, \\
&\phi_a(x) \longrightarrow \phi'_a(x') = D[\phi]_a(f^{-1}(x')) \simeq \phi_a(x') + \alpha^i \Delta_{ai}(\phi(x')),
\end{aligned}$$

- Show that charge Q_i built starting from the Noether's current is the generator of the transformation:

$$\delta_\alpha \phi_a(x) \equiv \phi'_a(x) - \phi_a(x) = \alpha^i \{Q_i, \phi_a(x)\} = \alpha^i \Delta_{ai}(x).$$

- Compute explicitly the charges associated to spacetime translation and Lorentz transformations in a scalar field theory.

- Using the formalism of the Poisson brackets check that the charges obtained generate the respective infinitesimal transformations.