Exercise 1: Poincaré algebra on fields

Consider a scalar field φ(x). Show that the Lorentz generators are represented on scalar fields by

\[ L^{\mu \nu} = i(x^\mu \partial^\nu - x^\nu \partial^\mu), \]

and verify that the commutators of generators indeed respect the Lorentz algebra.

Introduce now translations: show that the representation of the generators \( P^\mu \) of translations on fields is

\[ P^\mu = i \partial^\mu. \]

Compute the commutators

\[ [P^\mu, P^\nu] = ?, \quad [P^\mu, L^{\rho \sigma}] = ? \]

using their explicit representation on fields (the algebra is the same in any representation).

Exercise 2: (0, 1/2) and (1/2, 0) representations of the Lorentz group.

Starting from the explicit form of the spin-1/2 representations of \( SU(2) \) determine the form of the Lorentz transformations in the (0, 1/2) and (1/2, 0) representations. (Hint: How are the generators of the Lorentz group represented in these representations?) What about the (1/2, 1/2) representation?

Exercise 3: Lorentz boosts and rapidity

Consider a Lorentz boost along the x-axis

\[ \Lambda = \exp[-i \eta J^{10}]. \]

The parameter \( \eta \) is called rapidity of the boost along that axis. Show that this boost can be written as

\[ \Lambda = \begin{pmatrix} \cosh(\eta) & -\sinh(\eta) & 0 & 0 \\ -\sinh(\eta) & \cosh(\eta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \]

Express the velocity \( \beta \) and the boost factor \( \gamma \) in terms of the rapidity.

Show that applying two boosts along the same direction, characterized by rapidities \( \eta \) and \( \eta' \), the total transformation is again a boost along the same direction, characterized by rapidity \( \eta + \eta' \).

Exercise 4: Lorentz invariance of Lagrangians

Consider a real scalar field \( \phi(x) \). Write how the field \( \phi(x) \) transforms under the action of an element of the Poincaré group. Take the action \( S \) built starting from the following Lagrangian density

\[ S = \int dt d^3x \mathcal{L}, \quad \mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi). \]

Show that:

- the volume element \( dt d^3x \) is invariant under Poincaré transformations;
• the Poincaré group is a symmetry of the theory.

Repeat the analysis for the electromagnetic field:

• identify how the vector field $A_\mu$ transforms under the action of an element of the Poincaré group;
• show that the Poincaré group is a symmetry of the theory $\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$.

**Homework 1: Tensor product of $SU(2)$ vectors**

• Using the highest-weight technique reduce the tensor product of two spin-1 $SU(2)$ representations.
• An $SU(2)$ vector can be thought as a 1-index tensor $v_i$ transforming as $v_i \rightarrow \sum_j R_{ij}v_j$, where $R$ is an orthogonal matrix. Starting from the product of 2 vectors $v_i w_j$, show that this is a representation of $SU(2)$: the tensor product of 2 vector representations. Show that it is reducible and do the reduction in terms of irreducible tensor representations of $SU(2)$. Show that the result matches with what you found by the highest weight technique.

**Homework 2: Noether’s currents in classical mechanics**

Consider the general Lagrangian of a classical system

$$L = \sum_a \frac{m_a}{2}(\dot{q}_a)^2 - \sum_{a \neq b} V(|q_a - q_b|).$$

Using the Noether procedure compute the Noether current associated to the following transformations

• time translations;
• coordinate translations;
• coordinate rotations.