Quantum Field Theory

Set 7

Exercise 1: Poincaré algebra on fields

Consider a scalar field $\phi(x)$. Show that the Lorentz generators are represented on scalar fields by

$$L^{\mu\nu} = i(x^\mu \partial^\nu - x^\nu \partial^\mu),$$

and verify that the commutators of generators indeed respect the Lorentz algebra.
Introduce now translations: show that the representation of the generators P^μ of translations on fields is

$$P^\mu = i\partial^\mu.$$

Compute the commutators

$$[P^\mu, P^\nu] = ?, \qquad [P^\mu, L^{\rho\sigma}] = ?$$

using their explicit representation on fields (the algebra is the same in any representation).

Exercise 2: $(0, 1/2)$ and $(1/2, 0)$ representations of the Lorentz group.

Starting from the explicit form of the spin-1/2 representations of $SU(2)$ determine the form of the Lorentz transformations in the $(0, 1/2)$ and $(1/2, 0)$ representations. (Hint: How are the generators of the Lorentz group represented in these representations?). What about the $(1/2, 1/2)$ representation?

Exercise 3: Lorentz boosts and rapidity

Consider a Lorentz boost along the x-axis

$$\Lambda = \exp[-i\eta \mathcal{J}^{10}].$$

The parameter η is called *rapidity* of the boost along that axis. Show that this boost can be written as

$$\Lambda = \begin{pmatrix} \cosh(\eta) & -\sinh(\eta) & 0 & 0 \\ -\sinh(\eta) & \cosh(\eta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Express the velocity β and the boost factor γ in terms of the rapidity.
Show that applying two boosts along the same direction, characterized by rapidities η and η', the total transformation is again a boost along the same direction, characterized by rapidity $\eta + \eta'$.

Exercise 4: Lorentz invariance of Lagrangians

Consider a real scalar field $\phi(x)$. Write how the field $\phi(x)$ transforms under the action of an element of the Poincaré group. Take the action S built starting from the following Lagrangian density

$$S = \int dt\, d^3x\ \mathcal{L}, \qquad \mathcal{L} = \frac{1}{2}\partial_\mu \phi\, \partial^\mu \phi - V(\phi).$$

Show that:

- the volume element dtd^3x is invariant under Poincaré transformations;

- the Poincaré group is a symmetry of the theory.

Repeat the analysis for the electromagnetic field:

- identify how the vector field A_μ transforms under the action of an element of the Poincaré group;
- show that the Poincaré group is a symmetry of the theory $\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}$.

Homework 1: Tensor product of $SU(2)$ vectors

- Using the highest-weight technique reduce the tensor product of two spin-1 $SU(2)$ representations.
- An $SU(2)$ vector can be thought as a 1-index tensor v_i transforming as $v_i \to \sum_j R_{ij} v_j$, where R is an orthogonal matrix. Starting from the product of 2 vectors $v_i w_j$, show that this is a representation of $SU(2)$: the tensor product of 2 vector representations. Show that it is reducible and do the reduction in terms of irreducible tensor representations of $SU(2)$. Show that the result matches with what you found by the highest weight technique.

Homework 2: Noether's currents in classical mechanics

Consider the general Lagrangian of a classical system

$$L = \sum_a \frac{m_a}{2} (\dot{\vec{q}}_a)^2 - \sum_{a \neq b} V\left(|\vec{q}_a - \vec{q}_b|\right) .$$

Using the Noether procedure compute the Noether current associated to the following transformations

- time translations;
- coordinate translations;
- coordinate rotations.