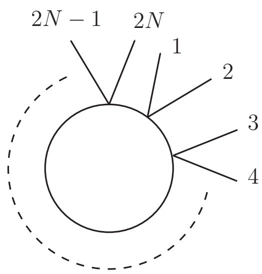
Advanced Quantum Field Theory

Exercise 7

Consider a real massive scalar field with quartic interaction:

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \tag{1}$$

Show that the 1-loop diagram with $2N$ external legs has the value (including the combinatoric factor):



$$\sim \frac{(2N)!}{2^{N+1}N} \quad (-i\lambda)^N \int \frac{d^d q}{(2\pi)} \frac{1}{(q^2 - m^2)^N}$$

Write the quantum effective action as an infinite series of diagrams:

$$\Gamma[\phi_{cl}] = \sum_m \frac{1}{(2m)!}\Gamma^{(2m)}\phi_{cl}^2 m \tag{2}$$

where

$$\Gamma^{(2m)} = \frac{\delta^{2m}\Gamma}{\delta\phi_{cl}\cdots\delta\phi_{cl}} = \text{1PI n-points correlation function} \tag{3}$$

Recalling the expression for the effective potential computed with the determinant-formalism:

$$V_{\text{eff}}[\phi_{cl}] = -\frac{1}{VT}\Gamma[\phi_{cl}]\Big|_{\phi_{cl}=\text{const}} = -\frac{1}{4(4\pi)^2}\left(m^2 + \frac{\lambda}{2}\phi_{cl}^2\right)^2\left(\frac{3}{2} + \log\frac{\tilde{\Lambda}}{m^2 + \frac{\lambda}{2}\phi_{cl}^2}\right) + \text{counter terms} \tag{4}$$

expand $V_{\text{eff}}[\phi_{cl}]$ in powers of ϕ_{cl} and verify that

$$\Gamma[\phi_{cl}]\big|_{\phi_{cl}=const} = -V_{\text{eff}}[\phi_{cl}] \tag{5}$$

Consider now the scalar QED with quartic self-interaction:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + D_\mu\phi^\dagger D^\mu\phi - \frac{\lambda}{4!}(|\phi|^2)^2 \tag{6}$$

where the covariant derivative is defined as $D_\mu\phi = \partial_\mu\phi + ieA_\mu\phi$. Expand the Lagrangian in terms of ϕ_1 and ϕ_2, where

$$\phi = \frac{\phi_1 + i\phi_2}{\sqrt{2}} \tag{7}$$

and extract the Feynman rules involving ϕ_1. Consider the infinite series

$$\tilde{\Gamma}[\phi_{1cl}] = \sum_m \frac{1}{(2m)!}\tilde{\Gamma}^{(2m)}\phi_{1cl}^2 m \tag{8}$$

where $\tilde{\Gamma}^{(2m)}$ are the 1-loop 1PI correlation function with $2m$ ϕ_1-external fields. Since we only want to compute the quantum effective potential we can set $\phi_{cl} = const$. This resembles in considering vanishing external momenta in all the correlation functions. As a all consequence 1-loop diagrams involving cubic vertices vanishes identically in the Landau gauge. Why?

Compute $\tilde{\Gamma}^{(2m)}$ and resum the series. Restore the ϕ_{2cl} dependence substituting $\phi_{1cl}^2 \to 2|\phi|^2$. What is the symmetry argument that make the latter substitution produce the right result?