Quantum Field Theory

Set 6

Exercise 1: Tensor product of $SU(2)$ representations

The statement that two particles have spin 1/2 reflects the fact that each particle can exist in two different states corresponding to the values of the z-component of the angular momentum. These two states define a vector space on which the group $SU(2)$ is represented. Consider the total spin of the bound state of the two particles: this corresponds to taking the tensor product of two $j = 1/2$ representations. Show that the resulting vector space,

$$V = \left\{ |s_1\rangle \otimes |s_2\rangle \,,\ s_i = \pm\frac{1}{2} \right\},$$

can be decomposed in a direct sum of two spaces on which act irreducible representations of $SU(2)$. Find these representations.

Exercise 2: Lorentz Group and Lorentz Algebra

Consider the set of the Lorentz transformations in space-time. Show that they correspond to the group

$$SO(1,3) = \left\{ \Lambda \in GL(4,\mathbb{R}) \mid \Lambda^T \eta \Lambda = \eta \right\},$$

where η is the metric with Minkowski signature $\eta = \mathrm{diag}(1,-1,-1,-1)$. Starting from the above definition

- identify the Lie algebra;
- compute the dimension of the Lie algebra;
- show that the following expression represents a basis of the algebra
$$(\mathcal{J}^{\mu\nu})^{\rho}{}_{\sigma} = i\left(\eta^{\mu\rho}\delta^{\nu}_{\sigma} - \eta^{\nu\rho}\delta^{\mu}_{\sigma}\right);$$
- show that the Lie algebra is
$$\left[\mathcal{J}^{\mu\nu}, \mathcal{J}^{\alpha\beta}\right] = i(\eta^{\nu\alpha}\mathcal{J}^{\mu\beta} - \eta^{\mu\alpha}\mathcal{J}^{\nu\beta} + \eta^{\mu\beta}\mathcal{J}^{\nu\alpha} - \eta^{\nu\beta}\mathcal{J}^{\mu\alpha});$$
- define the quantities
$$J^i = \frac{1}{2}\epsilon^{ijk}\mathcal{J}^{jk}\,, \qquad K^i = \mathcal{J}^{i0}\,,$$
and compute the commutation relations between them:
$$\left[J^i, J^j\right] = ?\,, \qquad \left[J^i, K^j\right] = ?\,, \qquad \left[K^i, K^j\right] = ?\,.$$

Exercise 3: Representation of Poincaré Group on functions

Call H the set of functions of a four-vector x^μ and denote $\phi(x)$ a generic element of it. Take an element of the Poincaré group acting on x^μ as:

$$g \equiv (\Lambda, a) :\ x^\mu \longrightarrow \Lambda^\mu{}_\nu\, x^\nu + a^\mu \ \equiv P_{(\Lambda,a)}(x^\mu).$$

Define the action of g on H as follows

$$\mathcal{D}_{(\Lambda,0)}[\phi](x) = \phi(\Lambda^{-1}x)\,, \qquad \mathcal{D}_{(0,a)}[\phi](x) = \phi(x-a)\,,$$
$$\mathcal{D}_{(\Lambda,a)}[\phi](x) = \mathcal{D}_{(0,a)}\left[\mathcal{D}_{(\Lambda,0)}[\phi]\right](x) = \phi(\Lambda^{-1}(x-a)).$$

Show that this definition respects the group composition law and defines therefore a representation.