Exercise 1: Tensor product of $SU(2)$ representations

The statement that two particles have spin $1/2$ reflects the fact that each particle can exist in two different states corresponding to the values of the $z$-component of the angular momentum. These two states define a vector space on which the group $SU(2)$ is represented. Consider the total spin of the bound state of the two particles: this corresponds to taking the tensor product of two $j = 1/2$ representations. Show that the resulting vector space,

$$ V = \left\{ |s_1\rangle \otimes |s_2\rangle , \; s_i = \pm \frac{1}{2} \right\}, $$

can be decomposed in a direct sum of two spaces on which act irreducible representations of $SU(2)$. Find these representations.

Exercise 2: Lorentz Group and Lorentz Algebra

Consider the set of the Lorentz transformations in space-time. Show that they correspond to the group $SO(1,3) = \{ \Lambda \in GL(4,\mathbb{R}) \mid \Lambda^T \eta \Lambda = \eta \}$, where $\eta$ is the metric with Minkowski signature $\eta = \text{diag}(1,-1,-1,-1)$. Starting from the above definition

- identify the Lie algebra;
- compute the dimension of the Lie algebra;
- show that the following expression represents a basis of the algebra
  $$ (\mathcal{J}^{\mu\nu})^\rho_\sigma = i (\eta^{\mu\rho} \delta^\nu_\sigma - \eta^{\nu\rho} \delta^\mu_\sigma); $$
- show that the Lie algebra is
  $$ [\mathcal{J}^{\mu\nu}, \mathcal{J}^{\alpha\beta}] = i (\eta^{\mu\alpha} \mathcal{J}^{\nu\beta} - \eta^{\nu\alpha} \mathcal{J}^{\mu\beta} + \eta^{\mu\beta} \mathcal{J}^{\nu\alpha} - \eta^{\nu\beta} \mathcal{J}^{\mu\alpha}); $$
- define the quantities
  $$ J^i = \frac{1}{2} \epsilon^{ijk} \mathcal{J}^{jk}, \quad K^i = \mathcal{J}^{i0}, $$
  and compute the commutation relations between them:
  $$ [J^i, J^j] = ?, \quad [J^i, K^j] = ?, \quad [K^i, K^j] = ?. $$

Exercise 3: Representation of Poincaré Group on functions

Call $H$ the set of functions of a four-vector $x^\mu$ and denote $\phi(x)$ a generic element of it. Take an element of the Poincaré group acting on $x^\mu$ as:

$$ g \equiv (\Lambda, a) : \; x^\mu \longrightarrow \Lambda^\mu_{\nu} x^\nu + a^\mu \equiv P_{(\Lambda,a)}(x^\mu). $$

Define the action of $g$ on $H$ as follows

- $D_{(\Lambda,0)}[\phi](x) = \phi(\Lambda^{-1} x)$,
- $D_{(0,a)}[\phi](x) = \phi(x - a)$,
- $D_{(\Lambda,a)}[\phi](x) = D_{(0,a)}[D_{(\Lambda,0)}[\phi]](x) = \phi(\Lambda^{-1}(x-a))$.

Show that this definition respects the group composition law and defines therefore a representation.