Quantum Field Theory

Set 5

Exercise 1: Lie Algebra

Consider a Lie group G parametrized by $(\alpha_1, ..., \alpha_n) = \alpha \in \mathbb{R}^n$ and a representation D defined by

$$g(\alpha) \to D(g(\alpha)),$$
$$e = g(0) \to \mathbb{1}.$$

Use the shorthand notation $D(\alpha) \equiv D(g(\alpha))$. Expand $D(\alpha)$ up to the linear term: $D(\alpha) \simeq \mathbb{1} + i\alpha_k X^k + O(\alpha^2)$ in a neighborhood of the identity so that the $k-$th generator in the representation D is $X^k \equiv -i\frac{\partial}{\partial \alpha_k} D(\alpha)\Big|_{\alpha=0}$.

- Starting from the product of two matrices $D(\alpha)\ D(\beta) = D(g(\alpha) \circ g(\beta)) \equiv D(p(\alpha, \beta))$ show that

$$iT_m(\alpha) \equiv D(\alpha)^{-1} \frac{\partial D}{\partial \alpha^m} = iX_j (\mathcal{U}^{-1})^j_{\ m}\,, \qquad \mathcal{U}^m_j \equiv \frac{\partial p^m(\alpha, \beta)}{\partial \beta^j}\Big|_{\beta=0}.$$

- Starting from the expression

$$i\left(\frac{\partial T_j(\alpha)}{\partial \alpha_k} - \frac{\partial T_k(\alpha)}{\partial \alpha_j}\right),$$

  and applying first the definition of $T_m(\alpha)$ and then the equality proved in the first point of the exercise show that the matrices X^k together with the usual commutator $[,]$ form a Lie Algebra.

Exercise 2: $SO(3)$ vs $SU(2)$

Consider the representation of the $SO(3)$ group on the three dimensional vector space $\mathbb{R}^3$.

- Show that the set of three 3×3 matrices $(T^a)_i^{\ j} = -i\epsilon_{aij}$ $(a = 1, 2, 3)$ is a basis for (a representation of) the Lie Algebra of $SO(3)$, and find this algebra explicitly (i.e. find the structure constants of $so(3)$).
- Consider an element of the group $R(\vec{\alpha}) = e^{i\alpha^a T^a}$. Write $\vec{\alpha} = \theta\,\vec{n}$ with $\vec{n} = \vec{\alpha}/|\vec{\alpha}|$. Expanding for $\theta << 1$, find how a vector $\vec{x} = (x^1, x^2, x^3) \in \mathbb{R}^3$ transforms under the action of an infinitesimal element of the group ($R(\vec{\alpha})$ represents a rotation around the $\vec{n}$ direction by an angle θ).

Take the group $SU(2)$. Consider the representation of this group on the following vector space:

$$V = \left\{M \in M(2 \times 2, \mathbb{C}) | M = M^\dagger,\ \mathrm{Tr}[M] = 0\right\}.$$

The action of an element U of the group (recall that U is itself a 2×2 matrix) is given by

$$U : M \to UMU^\dagger.$$

- Show that the three Pauli matrices σ^i form a basis of the space V and therefore any $M \in V$ can be written as $M = \sum_{i=1}^3 y_i \sigma^i$.
- Compute all the commutators between the Pauli matrices.
- Show that the action $U : M \to UMU^\dagger$ of $SU(2)$ really defines a representation.
- Show that the action of a given element U of $SU(2)$ corresponds to a rotation of the three dimensional vector $\vec{y} \equiv (y^1, y^2, y^3)$. Is this a faithful representation for $SU(2)$?

Exercise 3: Irreducible representations of $SU(2)$

Consider the Algebra of the group $SU(2)$

$$\left[T^a, T^b\right] = i\epsilon^{abc}T^c.$$

- Find all the commutation rules between the following quantities

$$T^{\pm} = \frac{T^1 \pm iT^2}{\sqrt{2}}; \qquad T^3.$$

- Show that the *Casimir operator* $J^2 \equiv (T^1)^2 + (T^2)^2 + (T^3)^2$ is proportional to the identity in any irreducible representation: $(\tau)^2 = \mu^2 \times 1_N$. (Here we denote as $(\tau)^2$ the representative of the operator J^2).

Consider an irreducible representation of the Algebra on a given vector space V. Denote $|m\rangle$ an eigenvector of the generator τ^3 relative to the eigenvalue m.

$$\tau^3|m\rangle = m|m\rangle.$$

- Compute the action of $\tau^{\pm}$ on $|m\rangle$.
- Prove that $|m| + m^2 \leq \mu^2$.
- Construct the irreducible representation which $|m\rangle$ belongs to.
- Compute the dimension of the representation and show that $\mu^2 = j(j+1)$ with j integer or semi-integer.
- Construct the representation corresponding to $j = 1/2$ and $j = 1$.

Exercise 4: Two exercises on representation theory

- Prove that a reducible finite-dimensional unitary representation is completely reducible (meaning that the orthogonal complement of its invariant subspace is itself invariant).
- Given two finite-dimensional inequivalent irreducible representations D_1 and D_2 of a group G on two vector spaces V_1 and V_2 show that the operators $D(g) \in Lin(V_1 \times V_2)$, $g \in G$ defined by

$$D(g)(v_1, v_2) = (D_1(g)v_1, D_2(g)v_2), \quad v_1 \in V_1, \ v_2 \in V_2 \tag{1}$$

furnish a (reducible) representation of G (called the direct sum of D_1 and D_2). Given an operator $A \in Lin(V_1 \times V_2)$ and $A \neq 0$, such that $AD(g) = D(g)A$ for every $g \in G$ show that $A(v_1, v_2)^T = (\lambda_1 v_1, \lambda_2 v_2)^T$, that is A is a multiple of the identity over both subspaces $(V_1, 0)$ and $(0, V_2)$ of $V_1 \times V_2$