Advanced Quantum Field Theory

Exercise 3

Consider a non-abelian gauge theory with gauge group $SU(N)$ and N_f Dirac fermions in the fundamental of $SU(N)$:

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\,\mu\nu} + \sum_{i=1}^{N_f} \bar{\psi}_k \,(i \not{D} - m_k\,)\, \psi_k + \mathcal{L}_{ghosts} + \mathcal{L}_{gauge\ fixing} \tag{1}$$

where we have defined the covariant derivative and the non-abelian field strength as

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu \tag{2}$$

$$(D_\mu \psi)_i = \partial \psi_i - i g A^a_\mu T^a_{ij} \psi_j \tag{3}$$

Note that in the above equation the indexes i, j runs over the fundamental representation of $SU(N)$ whereas the index k in the Lagrangian span the N_f different representations. The quantities T^a_{ij} are the generators of $SU(N)$ in the fundamental representation, satisfying the commutation relations:

$$[T^a, T^b] = i f^{abc} T^c \tag{4}$$

and normalized such that

$$\mathrm{Tr}[T^a T^b] = \frac{1}{2}\delta^{ab}\,, \qquad f^{abc} f^{dbc} = \frac{N^2-1}{2N}\delta^{ab} \tag{5}$$

Finally we use the gauge fixing and ghost Lagrangian

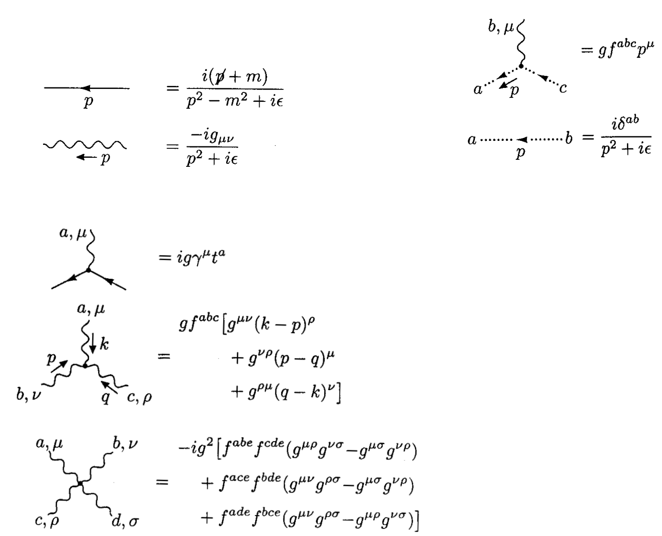
$$\mathcal{L}_{gauge\ fixing} = -\frac{1}{2\xi}(\partial^\mu A^a_\mu)^2 \tag{6}$$

$$\mathcal{L}_{ghost} = \bar{c}^a \partial^\mu (D_\mu c)^a \tag{7}$$

The ghosts $\bar{c}^a$, c^a are scalar particles with fermionic statistic (hence they anticommute) transforming in the adjoint representation of $SU(N)$. Their covariant derivatives reads:

$$(D_\mu c)^a = \partial_\mu c^a + g f^{abc} A^b_\mu c^c \tag{8}$$

From the above Lagrangian one can extract the Feynman rules for this theory (in the Feynman gauge $\xi = 1$):



Compute the 1PI two-point function of two gluon fields.
(Recall that a fermion loop and or a ghost loop has an additional (-1) factor).

Compute the 1PI of two fermionic fields (fermion self energy)

Compute the 1PI correlation functions of 2 fermions and 1gluon (vertex correction).

Express the Lagrangian in terms of renormalized fields:

$$A^a_\mu = Z_A^{1/2} A^{Ra}_\mu \qquad \psi = Z_\psi^{1/2} \psi^R \tag{9}$$

and define the renormalized coupling g_R by the relation

$$g_R = Z_A^{1/2} Z_\psi g - g_R \Delta_V \tag{10}$$

where Δ_V is the counter-term needed to cancel the divergence in the vertex containing two fermions and one gluon:

$$\Delta\mathcal{L} \supset g_R \Delta_V \bar{\psi}_R \not{A} \psi \tag{11}$$

Find the expression of Z_A , Z_ψ , Δ_V in the $\overline{\mathrm{MS}}$ renormalization scheme.

Compute the expression of the β function of the gauge coupling in terms of the above counter-terms. Compute explicitly β .