Quantum Field Theory

Set 22

Exercise 1: Application of Lippman-Schwinger equation

Starting from the usual Lippman-Schwinger equation,

$$|\psi_\alpha^+\rangle = |\phi_\alpha\rangle + \frac{1}{E_\alpha - H_0 + i\epsilon} H_I |\psi_\alpha^+\rangle,$$

show that it can be written in the equivalent form

$$|\psi_\alpha^+\rangle = |\phi_\alpha\rangle + \frac{1}{E_\alpha - H + i\epsilon} H_I |\phi_\alpha\rangle.$$

Use this formula to deduce the T-matrix element $T_{\beta\alpha} \equiv \langle\phi_\beta|H_I|\psi_\alpha^+\rangle \equiv \langle\phi_\beta|T|\phi_\alpha\rangle$ in the case in which $H_I = V_1(\vec{x}) + V_2(\vec{x})$, with $V_2(\vec{x}) = V_1(\vec{x} + \vec{A}) \equiv e^{i\vec{P}\cdot\vec{A}} V_1(\vec{x}) e^{-i\vec{P}\cdot\vec{A}}$, where $V_1(\vec{x})$ is significantly different from 0 only in a small region around a given point $\vec{x}_0$, and the distance $|\vec{A}|$ between the two potentials is much larger than the linear size of that region. Show that in this limit the T-matrix splits up in two independent pieces, $T = T_1(\vec{x}) + e^{i\vec{P}\cdot\vec{A}} T_1(\vec{x}) e^{-i\vec{P}\cdot\vec{A}}$.
Generalize this formula to an interaction Hamiltonian made of a set of N potentials $V_j(\vec{x})$ (all mutually far apart), and to the continuum case.

Exercise 2: Differential cross section $2 \to 2$ in the center of mass

Consider a scattering process $AB \to CD$ where the particles have four-momenta satisfying $P_A + P_B = P_C + P_D$. The differential cross section reads

$$d\sigma_{AB\to CD} = \frac{1}{4E_A E_B |\vec{v}_A - \vec{v}_B|} \left|\mathcal{M}_{AB\to CD}\right|^2 d\Phi_2\,,$$

where the 2-bodies phase space is

$$d\Phi_2 = \frac{d^3p_C}{(2\pi)^3 2E_C} \frac{d^3p_D}{(2\pi)^3 2E_D} (2\pi)^4 \delta^4\left(P_A + P_B - P_C - P_D\right).$$

Show that for particles colliding along the $\hat{z}$ axis the flux factor can be written as

$$E_A E_B |v_A^z - v_B^z| = \sqrt{(P_A^\mu P_{B\mu})^2 - m_A^2 m_B^2}.$$

If we are dealing with scalar particles in the initial and final state or we sum and average over all possible polarizations of non-scalar particles the Lorentz invariant matrix element $\left|\mathcal{M}_{AB\to CD}\right|^2$ can only depend on scalar combinations of the four momenta P_i. In general it could depend also on the polarizations of initial or final states. Introduce thus the *Mandelstam variables*

$$s = (P_A + P_B)^2 = (P_C + P_D)^2, \qquad t = (P_A - P_C)^2 = (P_B - P_D)^2, \qquad u = (P_A - P_D)^2 = (P_B - P_C)^2,$$

and compute the two bodies phase space and the differential cross section in the center of mass frame. In doing that, show that the scattering process has two degrees of freedom, which can be chosen as the polar and azimuthal angles of the scattering products w.r.t. the incoming particles. Express the phase space, and consequently the differential cross section, in terms of these variables, the masses, and the Mandelstam invariant s.
Write down the expression for the cross section in the particular cases $m_C = m_D = m$ or $m_C = 0$, $m_D = m$.