Quantum Field Theory

Set 21

Exercise 1: Invariance of Weyl fermion Lagrangian under time-reversal

Recalling the action of the *anti-unitary* operator of time-reversal U_T on Weyl fermions,

$$U_T^\dagger \, \chi_L^\alpha(t,\vec{x}) \, U_T = \eta_T \, \epsilon^{\alpha\beta} \chi_L^\beta(-t,\vec{x}),$$

show explicitly that the following Lagrangian is invariant:

$$\mathcal{L}_W = i\chi_L^\dagger \, \bar{\sigma}^\mu \partial_\mu \, \chi_L,$$

where, as usual, $\bar{\sigma}^\mu = (\mathbb{1}_2, -\sigma^i)$.

Exercise 2: Asymptotic states in Quantum Mechanics

Consider a onedimensional quantum system with a potential which is significantly different from zero only in a region $x \in [-L, L]$ and is rapidly decreasing outside. Call $\psi_k(x)$ the solution (of the Schroedinger equation for the interacting theory) corresponding to an energy $E_k = \frac{k^2}{2m}$. In the region where the potential is approximately zero we can write a particular solution with energy E_k as

$$\psi_k^+(x) = \begin{cases} e^{ikx} + Re^{-ikx}, & x \ll -L, \\ Te^{ikx}, & x \gg L, \end{cases}$$

where R and T are coefficients. Consider now a wave packet $\psi^+(x)$ which is a (narrow) gaussian superposition of particular solutions $\psi_k^+(x)$ around a momentum p. Evolve $\psi^+(x)$ in the past ($t \to -\infty$) and show that the result describes a free wave-packet incoming from the left. Conclude that $\psi^+(x)$ is the in- asymptotic state.
Consider now another particular solution

$$\psi_k^-(x) = \begin{cases} e^{ikx} + R'e^{-ikx}, & x \gg L, \\ T'e^{ikx}, & x \ll -L, \end{cases}$$

and show analogously that the gaussian packet $\psi^-(x)$ is the out- asymptotic state, namely at $t \to \infty$ it describes a free packet escaping towards the right.

Exercise 3: Scattering from a general potential

Consider a state $|\phi_k\rangle$ which is an eigenstate of a free Hamiltonian H_0. For simplicity let us consider $H_0 = \frac{p^2}{2m}$. Let us assume that at a certain finite time t and a finite distance L the states start interacting with a potential V. The system is now described by the full Hamiltonian $H = H_0 + V$. We also assume that the interaction with the potential is localized in time and space, so that the system, far away from the interaction point and after enough time, can still be described in terms of eigenstates of H_0.
Recalling the Lippmann-Schwinger equation,

$$|\Psi_k^\pm\rangle = |\phi_k\rangle + \frac{1}{E_k - H_0 \pm i\epsilon} H_I |\Psi_k^\pm\rangle,$$

extract an expression for the asymptotic states for this framework and show that at distances much larger than the typical size of the interaction it holds:

$$\Psi_k^\pm(x) = e^{i\vec{k}\cdot\vec{x}} + \frac{e^{\pm ikr}}{r} f(\hat{x}, k).$$