Quantum Field Theory

Set 20

Exercise 1: Charge conjugation properties of creation operators

Given the expansion for a complex scalar

$$\phi(x) = \int d\Omega_{\vec{k}} \left[e^{-ikx} a(\vec{k}) + e^{ikx} b^\dagger(\vec{k}) \right],$$

and the transformation under charge conjugation $C\phi(x)C = \eta_C \phi^\dagger(x)$, derive the action of the charge conjugation operator C on $a(\vec{k})$ and $b(\vec{k})$:

$$Ca(\vec{k})C = \eta_C b(\vec{k}), \qquad Cb(\vec{k})C = \eta_C^* a(\vec{k}).$$

Exercise 2: Charge conjugation properties of a particle-antiparticle system

Consider a scalar particle-antiparticle pair in the center of mass frame. Assume that their total angular momentum is l. Hence this state can be written as

$$|\Phi_l\rangle = \int d\Omega_{\vec{p}}\ f_l(\vec{p}, -\vec{p}) a^\dagger(\vec{p}) b^\dagger(-\vec{p})|0\rangle.$$

Recalling the symmetry properties of a state with angular momentum l, namely $f_l(\vec{p}, -\vec{p}) = (-1)^l f_l(-\vec{p}, \vec{p})$, and the action of charge conjugation on scalars

$$Ca^\dagger(\vec{k})C = \eta_C^* b^\dagger(\vec{k}), \qquad Cb^\dagger(\vec{k})C = \eta_C a^\dagger(\vec{k}),$$

find the transformation properties of the state $|\Phi_l\rangle$ under C.
Consider now a generic state composed of a fermionic particle-antiparticle pair with angular momentum l and total spin S:

$$|\Psi_{l,S}\rangle = \sum_{r,t=1}^{2} \int d\Omega_{\vec{p}}, f_l(\vec{p}, -\vec{p})\, \chi_S(r,t)\, b^\dagger(r, \vec{p}) \tilde{d}^\dagger(t, -\vec{p})|0\rangle,$$

where $\tilde{d}^\dagger(t, -\vec{p}) \equiv d^\dagger(t', -\vec{p})\epsilon^{tt'}$. The action of charge conjugation is defined as

$$Cb^\dagger(r, \vec{k})C = -\eta_C^* \tilde{d}^\dagger(r, \vec{k}), \qquad Cd^\dagger(r, \vec{k})C = \eta_C \tilde{b}^\dagger(r, \vec{k}),$$

and the wave functions satisfy

$$f_l(\vec{p}, -\vec{p}) = (-1)^l f_l(-\vec{p}, \vec{p}), \qquad \chi_S(r,t) = (-1)^{S+1}\chi_S(t,r).$$

Find the transformation properties of the state $|\Psi_{l,S}\rangle$ under C.

Exercise 3: Time Reversal of the scalar current

Compute the action of the anti-linear time reversal operator T on the scalar current:

$$TJ^\mu(\vec{x}, t)T = \eta^{\mu\mu} J^\mu(\vec{x}, -t), \qquad J_\mu = i(\phi^\dagger \partial_\mu \phi - \partial_\mu \phi^\dagger\, \phi).$$

Notice that in the expression above the indices μ are not summed: it's a compact notation that stands for:

$$TJ^\mu(\vec{x}, t)T = \begin{cases} J^0(\vec{x}, -t) & \text{if } \mu = 0, \\ -J^i(\vec{x}, -t) & \text{if } \mu = 1, 2, 3. \end{cases}$$

Exercise 4: Transformation properties of fermionic bilinears

Knowing the transformation properties of a Dirac fermion ψ under parity and charge conjugation,

$$P^\dagger \psi(t,\vec{x}) P = \eta_P \gamma^0 \psi(t,-\vec{x}),$$
$$C^\dagger \psi(t,\vec{x}) C = -i\eta_C \gamma^2 \psi^*(t,\vec{x}),$$

deduce the transformation properties of all the bilinears of the form $\bar{\psi}\Gamma\psi$, where Γ is an element of the usual basis

$$\Gamma = \{1_4, \gamma^5, \gamma^\mu, \gamma^\mu\gamma^5, \gamma^{\mu\nu}\}.$$