Quantum Field Theory

Set 2

Exercise 1: Inelastic Dark Matter

A central task of modern cosmology is to determine what the Universe is made of. A number of observations suggest that the bulk of the matter in the universe is not luminous and doesn't consist of compact objects made of ordinary baryonic matter, like planets, etc. A possible explanation is that weakly interacting massive particles exist copiously in the halo of our galaxy but only rarely interact with ordinary matter. Numerous experiments have been set up in an attempt to directly detect these particles. Two of them are CDMS and DAMA. They both utilize the fact that the Earth, moving in the galaxy, should feel a "wind" of Dark Matter (DM) particles. These particles can scatter with some target nucleus and information can be extracted measuring its recoil energy.

In order to study the kinematics of this process consider a DM particle X with mass M_X striking a nucleus N at rest with mass $M_N \sim \mathcal{A}$ GeV, where $\mathcal{A}$ is the atomic number. Assuming elastic scattering, compute the minimum energy that the nucleus can acquire after the scattering.

We call *inelastic Dark Matter* a class of DM models in which elastic scatterings with a nucleus are forbidden for some reason and the only allowed process is $X + N \to Y + N$ where Y represents an excited state of the DM with energy $M_Y = M_X + \delta$ in its rest frame, with $\delta \ll M_X, M_N$. Show that the incoming DM particle X must have an energy above a certain value $\tilde{E}_{\rm min}$ in order for the process to happen. When $\tilde{E} = \tilde{E}_{\rm min}$ compute the energy of the nucleus after the scattering. What is the main difference with the elastic case discussed above? In order to observe inelastic DM would you set an experiment to detect very small nuclei recoils or finite energy recoils?

Knowing that CDMS uses Germanium ($\mathcal{A} = 73$) while DAMA uses Iodine ($\mathcal{A} = 127$) and assuming that DM has an average velocity of 220 Km/s and a mass $M_X = 100$ GeV, show that there is a range of values for δ for which scattering processes can be observed by DAMA but not by CDMS.

Exercise 2

Consider an infinite system of particles located along the direction x at distance a from one another. Each particle has mass m and is connected to the neighbor masses by a spring of elastic constant k and vanishing rest length. The masses are allowed to move only in the direction y. Call y_i the displacement of the mass at $x = i \cdot a$.
1. Find the Lagrangian that describes the system and the relative equations of motion.
2. Find the Hamiltonian and the first order equations of motion. Verify that they give rise to the same second order equations of motion.
3. Perform the continuum limit in the Lagrangian ($a \to 0$) keeping the following quantities constant:

- $\mu = \frac{m}{a}$ = mass per unit length of the chain of oscillator,
- $Y = k\,a$ = Young elastic coefficient.

4. Perform the same limit in the equations of motion and verify that they can be obtained as Euler-Lagrange equations of the Action.
5. Find the general solution of the equations of motion.
6. What happens if in the initial discrete model each particle is also connected to the point ($x = ia$, $y = 0$) by a spring of frequency w'? How does the solution change in the continuum limit? (keep w'=constant in the limit)

Exercise 3: Classical magnetic moment

Given a set of N classical particles of masses m_i and charges e_i compute the magnetic moment of the system. Assuming the relation $e_i/m_i = e/m \;\forall\; i$, show that the result is proportional to the angular momentum of the system.