Advanced Quantum Field Theory

Exercise 2

Consider a real massive scalar field with quartic interaction:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \tag{1}$$

The general form of the quantum effective action is:

$$\Gamma[\phi_{\text{cl}}] = \int d^4x \left(\frac{1}{2}\mathcal{Z}[\phi_{\text{cl}}]\partial_\mu\phi_{\text{cl}}\partial^\mu\phi_{\text{cl}} - V_{\text{eff}}[\phi_{\text{cl}}] + \text{higher der.}\right) \tag{2}$$

One can show that at lowest order in $\hbar$, $\Gamma[\phi_{\text{cl}}]^{(0)} = \int d^4x\mathcal{L}[\phi_{\text{cl}}]$ while the first quantum correction is given by

$$\Gamma[\phi_{\text{cl}}]^{(1)} = \frac{i}{2}\text{Tr}\left[\log\left(-\frac{\delta^2 S_r}{\delta\phi_r^2}[\phi_{\text{cl}}]\right)\right] \tag{3}$$

where the subscript r stands for renormalized. During the lecture we found:

$$V_{\text{eff}}[\phi_{\text{cl}}] = -\frac{1}{VT}\Gamma[\phi_{\text{cl}}]\bigg|_{\phi_{\text{cl}}=\text{const}} = -\frac{1}{4(4\pi)^2}\left(m^2 + \frac{\lambda}{2}\phi_{\text{cl}}^2\right)^2\left(\frac{3}{2} + \log\frac{\tilde{\Lambda}}{m^2 + \frac{\lambda}{2}\phi_{\text{cl}}^2}\right) + \text{counter terms} \tag{4}$$

Compute

$$\frac{\delta^2\Gamma[\phi_{\text{cl}}]}{\delta\phi_{\text{cl}}(x_1)\delta\phi_{\text{cl}}(x_2)}\bigg|_{\phi_{\text{cl}}=\text{const}} = \int d^4x\left(-\mathcal{Z}[\phi_{\text{cl}}]\delta^4(x-x_1)\Box\delta^4(x-x_2) - \frac{\delta^2 V_{\text{eff}}}{\delta\phi_{\text{cl}}(x_1)\delta\phi_{\text{cl}}(x_2)}[\phi_{\text{cl}}]\right)\bigg|_{\phi_{\text{cl}}=\text{const}} \tag{5}$$

in Fourier space, isolate the piece associated to the kinetic term $\Box$ and determine $\mathcal{Z}[\phi_{\text{cl}}]$.