Quantum Field Theory

Set 19

Exercise 1: Fermi Lagrangian

Consider the Lagrangian of a massive vector field A_μ coupled to a Dirac fermion ψ:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}M^2 A_\mu A^\mu + i\bar{\psi}\not{D}\psi.$$

Derive the equations of motion for the massive vector. Solve them formally making a Fourier transform. Consider the low energy (large distance) solution for A_μ and show that

$$A_\mu(x) \simeq \frac{q}{M^2}\bar{\psi}(x)\gamma_\mu\psi(x).$$

Plug this solution into the equations of motion for the fermion ψ and thus show that the same result can be obtained from the so called *Fermi Lagrangian* that contains an interaction involving 4 fermion fields.

Exercise 2: Parity transformation properties of a particle-antiparticle system

Consider a scalar particle and antiparticle pair in their center of mass frame. Assume their total angular momentum to be l. Hence this state can be written as

$$|\Phi_l\rangle = \int d\Omega_{\vec{p}}\; f_l(\vec{p},-\vec{p})a^\dagger(\vec{p})b^\dagger(-\vec{p})|0\rangle.$$

Recalling the symmetry properties of a state with angular momentum l, i.e. $f_l(\vec{p},-\vec{p}) = (-1)^l f_l(-\vec{p},\vec{p})$, and the action of parity on scalars

$$P^\dagger a^\dagger(\vec{k})P = \eta_P a^\dagger(-\vec{k}), \qquad P^\dagger b^\dagger(\vec{k})P = \eta_P b^\dagger(-\vec{k}),$$

find the transformation properties of the state $|\Phi_l\rangle$ under P.

Consider now a generic state composed of a fermionic particle-antiparticle pair with angular momentum l and total spin S:

$$|\Psi_{l,S}\rangle = \sum_{r,t=1}^{2}\int d\Omega_{\vec{p}}\; f_l(\vec{p},-\vec{p})\,\chi_S(r,t)\,\tilde{d}^\dagger(\vec{p},r)b^\dagger(-\vec{p},t)|0\rangle,$$

where $\tilde{d}^\dagger(\vec{p},r) \equiv d^\dagger(\vec{p},r')\epsilon^{rr'}$, $r=1,2$ creates an antiparticle with spin $+1/2,-1/2$ respectively (not $-1/2,1/2$ as it would be for $d^\dagger(\vec{p},r)$). The action of parity is defined as

$$P^\dagger b^\dagger(\vec{k},r)P = \eta_P b^\dagger(-\vec{k},r), \qquad P^\dagger \tilde{d}^\dagger(\vec{k},t)P = -\eta_P \tilde{d}^\dagger(-\vec{k},t),$$

and the wave functions satisfy

$$f_l(\vec{p},-\vec{p}) = (-1)^l f_l(-\vec{p},\vec{p}), \qquad \chi_S(r,t) = (-1)^{S+1}\chi_S(t,r).$$

Find the transformation properties of the state $|\Psi_{l,S}\rangle$ under P.

Exercise 3: Charge conjugation of Dirac Lagrangian

Recalling the transformation properties of Weyl fermions under charge conjugation

$$C^\dagger \, \chi_L \, C = \eta_L \epsilon \chi_R^*,$$
$$C^\dagger \, \chi_R \, C = \eta_R \epsilon \chi_L^*,$$

show that the Dirac action

$$S = \int d^4x \, \left(i\chi_L^\dagger \, \bar{\sigma}^\mu \partial_\mu \, \chi_L + i\chi_R^\dagger \sigma^\mu \partial_\mu \chi_R - m(\chi_R^\dagger \chi_L + \chi_L^\dagger \chi_R) \right)$$

is invariant only for the choice $\eta_R^* \eta_L = -1$. Derive the matrix U_C that describes the transformation properties of a Dirac fermion according to the following formula:

$$C^\dagger \, \psi \, C = \eta_C \, U_C \, \bar{\psi}^T, \qquad \psi = \begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix}.$$

Show that $U_C = i\gamma^0\gamma^2$.

Exercise 4: Boost of a polarization vector

Consider a massive vector field of momentum $p^\mu = (E, 0, 0, p)$, with positive helicity, i.e. with $\varepsilon^\mu = \frac{1}{\sqrt{2}}(0, 1, i, 0)$. Perform a boost along the y direction. Write the polarization vector after the boost and decompose it explicitly in the basis of polarization vectors with definite helicity.