Quantum Field Theory

Set 18

Exercise 1: Equivalence of Hamiltonian and Lagrangian formalism for massive vectors

Consider the Lagrangian of a massive vector field. Compute the conjugate momenta of A_μ and show that $\Pi_0 = 0$. This means that A_0 is not a dynamical variable but can be expressed in terms of the other fields. Show that $A_0 = -\frac{1}{M^2}\partial_i \Pi^i$. Then show that the Hamiltonian is:

$$H = \int d^3x \left(\frac{1}{2}\Pi^i \Pi^i + \frac{1}{2M^2}(\partial_i \Pi^i)^2 + \frac{1}{4}F^{ij}F^{ij} + \frac{1}{2}M^2 A^i A^i \right).$$

Using the following commutation relations

$$\begin{aligned}
&\left[A_i(\vec{x},t),\, \Pi^j(\vec{y},t)\right] = i\delta_i^j \delta^3(\vec{x}-\vec{y}),\\
&\left[A_i(\vec{x},t),\, A_j(\vec{y},t)\right] = \left[\Pi^i(\vec{x},t),\, \Pi^j(\vec{y},t)\right] = \left[A_0(\vec{x},t),\, \Pi^j(\vec{y},t)\right] = 0,\\
&\left[A_i(\vec{x},t),\, A_0(\vec{y},t)\right] = -\frac{1}{M^2}\left[A_i(\vec{x},t),\, \partial_m \Pi^m(\vec{y},t)\right] = \frac{i}{M^2}\partial_i^{(x)}\delta^3(\vec{x}-\vec{y}),
\end{aligned}$$

show that the Hamilton equations of motion are equivalent to the Lagrange equations of motion.

Exercise 2: Generator of rotations

Consider the Lagrangian of a massive vector field. Show that the Noether current associated to Lorentz transformation is

$$J^\mu_{\alpha\beta} = -F^{\mu\rho}\left(\eta_{\rho\alpha}A_\beta + x_\alpha \partial_\beta A_\rho\right) + F^{\mu\rho}\left(\eta_{\rho\beta}A_\alpha + x_\beta \partial_\alpha A_\rho\right) + \left(\delta^\mu_\alpha x_\beta - \delta^\mu_\beta x_\alpha\right)\mathcal{L}.$$

Consider the component J^0_{ij} and thus find the Noether charge associated to rotations. Show that

$$J_k = \frac{1}{2}\epsilon_{ijk}\int J^0_{ij} d^3x = \epsilon_{ijk}\int d^3x \left(\Pi_i A_j - \Pi_m\, x_i \partial_j A_m\right).$$

Substitute the expansion of $\Pi_i(x)$ and $A_j(x)$ in terms of $a_i(k)$ and $a_i(k)^\dagger$ and show that the generator of rotations can be written as

$$J_k = i\epsilon_{ijk}\int d\Omega_{\vec{k}} \left\{ a_i(\vec{k})a_j^\dagger(\vec{k}) - a_m(\vec{k})\left(k_i \frac{\partial}{\partial k^j}\right) a_m^\dagger(\vec{k}) \right\}.$$