Exercise 1: Equivalence of Hamiltonian and Lagrangian formalism for massive vectors

Consider the Lagrangian of a massive vector field. Compute the conjugate momenta of $A_\mu$ and show that $\Pi_0 = 0$. This means that $A_0$ is not a dynamical variable but can be expressed in terms of the other fields. Show that $A_0 = -\frac{1}{M^2}\partial_i \Pi^i$. Then show that the Hamiltonian is:

$$H = \int d^3x \left( \frac{1}{2}\Pi^i \Pi^i + \frac{1}{2M^2}(\partial_i \Pi^i)^2 + \frac{1}{4}F^{ij}F^{ij} + \frac{1}{2}M^2 A^i A^i \right).$$

Using the following commutation relations

$$[A_i(\vec{x}, t), \Pi^j(\vec{y}, t)] = i\delta_i^j \delta^3(\vec{x} - \vec{y}),$$
$$[A_i(\vec{x}, t), A_j(\vec{y}, t)] = [\Pi^i(\vec{x}, t), \Pi^j(\vec{y}, t)] = [A_0(\vec{x}, t), \Pi^j(\vec{y}, t)] = 0,$$
$$[A_i(\vec{x}, t), A_0(\vec{y}, t)] = -\frac{1}{M^2} [A_i(\vec{x}, t), \partial_m \Pi^m(\vec{y}, t)] = i\frac{1}{M^2} \delta_i^j \partial_j \delta^3(\vec{x} - \vec{y}),$$

show that the Hamilton equations of motion are equivalent to the Lagrange equations of motion.

Exercise 2: Generator of rotations

Consider the Lagrangian of a massive vector field. Show that the Noether current associated to Lorentz transformation is

$$J^\mu_{\alpha\beta} = -F^{\mu\rho}(\eta_{\alpha\beta} A_\rho + x_\alpha \partial_\beta A_\rho) + F^{\mu\rho}(\eta_{\beta\rho} A_\alpha + x_\beta \partial_\alpha A_\rho) + \left(\delta_\mu^\alpha x_\beta - \delta_\mu^\beta x_\alpha\right) L.$$

Consider the component $J^0_j$ and thus find the Noether charge associated to rotations. Show that

$$J_k = \frac{1}{2} \epsilon_{ijk} \int J^0_j d^3x = \epsilon_{ijk} \int d^3x (\Pi_i A_j - \Pi_j x_i \partial_i A_m).$$

Substitute the expansion of $\Pi_i(x)$ and $A_j(x)$ in terms of $a_i(k)$ and $a_i(k)\dagger$ and show that the generator of rotations can be written as

$$J_k = i\epsilon_{ijk} \int d^3k \left\{ a_i(k)a_j^\dagger(k) - a_m(k)\left( k_i \partial_j \right) a_m^\dagger(k) \right\}.$$