Exercise 1: Physical observables

Consider the Gupta-Bleuler Lagrangian:

\[ \mathcal{L}_{GB} = -\frac{1}{2} (\partial_{\mu} A_{\nu})(\partial^{\mu} A^{\nu}). \]

- Compute the conserved momentum \( P_{\nu} \) through the Noether procedure.
- By working with the algebra of the ladder operators, show that \( P_{\nu} \) is a physical observable in the sense that:

\[ [L, P_{\nu}] \sim \partial_{\nu} L. \]

where \( L \equiv \partial^{\mu} A_{\mu} \).

Exercise 2: Propagator of the Gupta-Bleuler Lagrangian

Consider the Gupta-Bleuler Lagrangian with generic coefficient \( \xi \), in presence of an external source \( J_{\mu} \):

\[ \mathcal{L}_{GB} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{\xi}{2} (\partial_{\mu} A^{\mu})^2 + A_{\mu} J_{\mu} \]

- Find the EOM for \( A^{\mu} \), and write it in a following form

\[ \Pi^{\mu\nu} A_{\nu} = J_{\mu} \]

where \( \Pi^{\mu\nu} \) is a tensor dependent on \( \partial_{\mu} \) and \( \eta^{\mu\nu} \)

- Invert the EOM:

\[ A^{\mu}(x) = (\Pi^{-1})^{\mu\nu} J_{\nu}(x) \quad (1) \]

where \( (\Pi^{-1})^{\mu\nu} \) is called propagator.

(Hint: Decompose \( \Pi^{\mu\nu} \) into the orthogonal projectors\( P_{\mu\nu}^{L} = \frac{1}{\Box} \partial^\mu \partial^\nu \), \( P_{\mu\nu}^{T} = \eta^{\mu\nu} - \frac{1}{\Box} \partial^\mu \partial^\nu \))

Is this procedure possible for \( \xi = 0 \)?

- Specialize now to the case \( \xi = 1 \) and solve for the Green function of the theory \( G^{\mu\nu}(x) \):

\[ G^{\mu\nu}(x) = (\Pi^{-1})^\mu_\alpha \eta^{\alpha\nu} \delta^4(x) \quad (2) \]

Use the prescription for going around the poles at \( k^0 = \pm |\vec{k}| \) in order to have the Retarded green function

- Use now instead the Feynman prescription, which is obtained by the replacement \( k^2 \rightarrow k^2 + i\epsilon \) for \( \epsilon \rightarrow 0^+ \).

(Do not perform the integral over \( \vec{k} \) explicitly).
Exercise 3: Non-relativistic limit of the Klein - Gordon - Fock equation

Start from the Klein - Gordon - Fock equation

\[ (\partial_\mu \partial^\mu + m^2) \psi = 0, \] (3)

and find that in the non-relativistic limit \( \Delta E \ll m \) (where \( \Delta E \) is the classical energy, \( E = \Delta E + m \)) this equation reduces to the Schrodinger equation,

\[ i \partial_t \tilde{\psi} = -\frac{\nabla^2 \tilde{\psi}}{2m}, \]

where \( \psi = \exp(-imt)\tilde{\psi} \).

*Exercise 4: Klein - Gordon - Fock equation in the external electrostatic potential

Solve the Klein - Gordon - Fock equation

\[ (D_\mu D^\mu + m^2) \psi = 0, \quad D_\mu \equiv \partial_\mu + ieA_\mu, \] (4)

in the electrostatic field with

\[ A^0 = \varphi = \frac{Z|e|}{r}, \quad \vec{A} = 0, \]

and find relativistic corrections to the Hydrogen energy levels up to the order \( O(\alpha^4) \), where \( \alpha \) is the fine structure constant.