Quantum Field Theory

Set 16

Exercise 1: Physical observables

Consider the Gupta-Bleuler Lagrangian:

$$\mathcal{L}_{GB} = -\frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu).$$

- Compute the conserved momentum P_ν through the Noether procedure.
- By working with the algebra of the ladder operators, show that P_ν is a physical observable in the sense that:

  $$[L, P_\nu] \sim \partial_\nu L.$$

  where $L \equiv \partial^\mu A^-_\mu$.

Exercise 2: Propagator of the Gupta-Bleurer Lagrangian

Consider the Gupta-Bleurer Lagrangian with generic coefficient ξ, in presence of an external source J^μ:

$$\mathcal{L}_{GB} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{\xi}{2}(\partial_\mu A^\mu)^2 + A^\mu J_\mu$$

- Find the EOM for A^μ, and write it in a following form

  $$\Pi^{\mu\nu}A_\nu = J_\mu$$

  where $\Pi^{\mu\nu}$ is a tensor dependent on ∂_μ and $\eta^{\mu\nu}$
- Invert the EOM:

  $$A^\mu(x) = (\Pi^{-1})^{\mu\nu}J_\nu(x) \tag{1}$$

  where $(\Pi^{-1})^{\mu\nu}$ is called *propagator*.

  (*Hint:* Decompose $\Pi^{\mu\nu}$ into the orthogonal projectors $P_L^{\mu\nu} = \frac{1}{\Box}\partial^\mu\partial^\nu$, $P_T^{\mu\nu} = \eta^{\mu\nu} - \frac{1}{\Box}\partial^\mu\partial^\nu$)

  Is this procedure possible for $\xi = 0$?
- Specialize now to the case $\xi = 1$ and solve for the Green function of the theory $G^{\mu\nu}(x)$:

  $$G^{\mu\nu}(x) = (\Pi^{-1})^\mu{}_\alpha \eta^{\alpha\nu}\delta^4(x) \tag{2}$$

  Use the prescription for going around the poles at $k^0 = \pm|\vec{k}|$ in order to have the *Retarded* green function
- Use now instead the Feynman prescription, which is obtained by the replacement $k^2 \to k^2 + i\epsilon$ for $\epsilon \to 0^+$. (Do not perform the integral over $\vec{k}$ explicitly).

Exercise 3: Non-relativistic limit of the Klein - Gordon - Fock equation

Start from the Klein - Gordon - Fock equation

$$\left(\partial_\mu \partial^\mu + m^2\right) \psi = 0 \,, \tag{3}$$

and find that in the non - relativistic limit $\Delta E \ll m$ (where ΔE is the classical energy, $E = \Delta E + m$) this equation reduces to the Schrodinger equation,

$$i\partial_t \tilde{\psi} = -\frac{\nabla^2 \tilde{\psi}}{2m} \,,$$

where $\psi = \exp(-imt)\tilde{\psi}$.

*Exercise 4: Klein - Gordon - Fock equation in the external electrostatic potential

Solve the Klein - Gordon - Fock equation

$$\left(D_\mu D^\mu + m^2\right) \psi = 0 \,, \quad D_\mu \equiv \partial_\mu + ieA_\mu \,, \tag{4}$$

in the electrostatic field with

$$A^0 = \varphi = \frac{Z|e|}{r} \,, \quad \vec{A} = 0 \,,$$

and find relativistic corrections to the Hydrogen energy levels up to the order $\mathcal{O}(\alpha^4)$, where α is the fine structure constant.