Quantum Field Theory

Set 15

Exercise 1: Maxwell's equations and transverse components

- Show the the quantities $P_\perp^{ij} = \left(\delta^{ij} - \frac{\partial^i\partial^j}{\nabla^2}\right)$, $P_L^{ij} = \frac{\partial^i\partial^j}{\nabla^2}$ are orthogonal projectors on the space of three-dimensional vector fields:

$$(P_L)^{ij}(P_\perp)^{jk} = 0\,, \qquad (P_L)^{ij}(P_L)^{jk} = (P_L)^{ik}\,, \qquad (P_\perp)^{ij}(P_\perp)^{jk} = (P_\perp)^{ik}\,, \qquad P_L^{ij} + P_\perp^{ij} = \delta^{ij}$$

- Decompose the electric and magnetic field into longitudinal and transverse parts, $\vec{E} = \vec{E}_L + \vec{E}_\perp$, $\vec{B} = \vec{B}_L + \vec{B}_\perp$, where:

$$E_\perp^i = P_\perp^{ij} E^j\,, \qquad E_L^i \equiv P_L^{ij} E^j$$

  Using the Bianchi identity for the field strength, namely $\epsilon_{\mu\nu\rho\sigma}\partial^\mu F^{\rho\sigma} = 0$, show that the number of degrees of freedom encoded in $F_{\mu\nu}$ is **3**. In particular, prove that $\vec{B}_L = 0$, while $\vec{B}_\perp$ can be expressed in terms of $\vec{E}_\perp$ only. Finally, consider Maxwell's inhomogeneous equations to show that $\vec{E}_L$ is completely fixed by the charge density. Thus, the *number of dynamical degrees of freedom* is only **2**.

- Now solve the Maxwell equation for A^0 and substitute the solution into the remaining non trivial equations. Show that the result is a wave equation for the transverse components of the gauge potential and that the longitudinal component decouples completely.

Exercise 2: Energy momentum tensor

Consider the following Lagrangian:

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{\lambda}{2}(\partial_\rho A^\rho)^2.$$

- Find the energy momentum tensor using the standard procedure and show explicitly that it is conserved by imposing the equations of motion.
- Discuss the gauge transformation properties of $T^\mu_{\ \nu}$ for $\lambda = 0$. What about the associated charges?
- Improve the energy momentum tensor you found for $\lambda = 0$ by adding a term $F^{\mu\rho}\partial_\rho A_\nu$. Show that the new tensor is still conserved and gives rise to the same charges as before, but is now also symmetric, traceless and gauge invariant.
- What happens if $\lambda \neq 0$? Do currents and charges depend on λ?

Exercise 3: Coulomb gauge

Consider the Lagrangian of a massless vector field:

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}\ ,$$

where $F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu$.

- Compute the equations of motion.

- Compute the conjugate momenta Π^i. What about Π^0? Compare the result with the usual canonical commutation relations. Are they consistent? Rewrite the equations of motion for the 0-component in terms of Π^i.

- Show that, thanks to the gauge invariance of the theory ($A_\mu(x) \longrightarrow A_\mu(x) - \partial_\mu \Lambda(x)$), we can always impose the constraint $\vec{\nabla} \cdot \vec{A} = 0$ (i.e. we can always find a Λ such that $\vec{A}$ satisfies that constraint).

- Consider the commutation relation:

$$[\Pi(\vec{x}, t)_j, A(\vec{y}, t)_i] = +i\delta_{ij}\delta^3(\vec{x} + \vec{y}) - i\partial_i^{(x)}\partial_j^{(y)}\frac{1}{4\pi|\vec{x} - \vec{y}|}.$$

  Show that the above expression is consistent with the gauge choice $\vec{\nabla} \cdot \vec{A} = 0$ and with the constraint $\vec{\nabla} \cdot \vec{\Pi} = 0$.