Exercise 1: Maxwell’s equations and transverse components

- Show the the quantities $P_{ij}^\perp = (\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2})$, $P_{ij}^L = \frac{\partial_i \partial_j}{\nabla^2}$ are orthogonal projectors on the space of three-dimensional vector fields:

  $$(P_L)^ij(P_L)^jk = 0, \quad (P_L)^ij(P_L)^ik = (P_L)^{ik}, \quad (P_{\perp})^ij(P_{\perp})^{jk} = (P_{\perp})^{ik}, \quad P_{ij}^L + P_{ij}^\perp = \delta^{ij}$$

- Decompose the electric and magnetic field into longitudinal and transverse parts, $\vec{E} = \vec{E}_L + \vec{E}_\perp$, $\vec{B} = \vec{B}_L + \vec{B}_\perp$, where:

  $E_i^\perp = P_{ij}^\perp E_j^i$, $E_i^L = P_{ij}^L E_j^i$

Using the Bianchi identity for the field strength, namely $\epsilon_{\mu \nu \rho \sigma} \partial^\mu F_{\rho \sigma} = 0$, show that the number of degrees of freedom encoded in $F_{\mu \nu}$ is 3. In particular, prove that $\vec{B}_L = 0$, while $\vec{B}_\perp$ can be expressed in terms of $\vec{E}_\perp$ only. Finally, consider Maxwell’s inhomogeneous equations to show that $\vec{E}_L$ is completely fixed by the charge density. Thus, the number of dynamical degrees of freedom is only 2.

- Now solve the Maxwell equation for $A^0$ and substitute the solution into the remaining non trivial equations. Show that the result is a wave equation for the transverse components of the gauge potential and that the longitudinal component decouples completely.

Exercise 2: Energy momentum tensor

Consider the following Lagrangian:

$$\mathcal{L} = -\frac{1}{4} F^{\mu \nu} F_{\mu \nu} - \frac{\lambda}{2} (\partial_\rho A^\rho)^2.$$

- Find the energy momentum tensor using the standard procedure and show explicitly that it is conserved by imposing the equations of motion.
- Discuss the gauge transformation properties of $T^\mu_\nu$ for $\lambda = 0$. What about the associated charges?
- Improve the energy momentum tensor you found for $\lambda = 0$ by adding a term $F^{\mu \rho} \partial_\rho A_\nu$. Show that the new tensor is still conserved and gives rise to the same charges as before, but is now also symmetric, traceless and gauge invariant.
- What happens if $\lambda \neq 0$? Do currents and charges depend on $\lambda$?

Exercise 3: Coulomb gauge

Consider the Lagrangian of a massless vector field:

$$\mathcal{L} = -\frac{1}{4} F^{\mu \nu} F_{\mu \nu},$$

where $F^{\mu \nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$.

- Compute the equations of motion.
• Compute the conjugate momenta Π\textsuperscript{i}. What about Π\textsuperscript{0}? Compare the result with the usual canonical commutation relations. Are they consistent? Rewrite the equations of motion for the 0-component in terms of Π\textsuperscript{i}.

• Show that, thanks to the gauge invariance of the theory (A_\textsubscript{µ}(x) \rightarrow A_\textsubscript{µ}(x) - \partial_\textsubscript{µ}\Lambda(x)), we can always impose the constraint \( \vec{\nabla} \cdot \vec{A} = 0 \) (i.e. we can always find a \( \Lambda \) such that \( \vec{A} \) satisfies that constraint).

• Consider the commutation relation:

\[
[\Pi(\vec{x}, t), A(\vec{y}, t)] = +i\delta_{ij}\delta^3(\vec{x} + \vec{y}) - i\partial_i^{(x)}\partial_j^{(y)} \frac{1}{4\pi|\vec{x} - \vec{y}|}.
\]

Show that the above expression is consistent with the gauge choice \( \vec{\nabla} \cdot \vec{A} = 0 \) and with the constraint \( \vec{\nabla} \cdot \vec{Π} = 0 \).