Quantum Field Theory

Set 14

Exercise 1: U(1) symmetry and chiral symmetry

Given the Lagrangian density of a massless Dirac fermion $\psi \equiv \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$:

$$\mathcal{L} = i\bar{\psi}\,\partial\!\!\!/\psi,$$

verify that it is invariant under a symmetry $U(1)_L \times U(1)_R$ where each $U(1)$ acts independently on the left or right component of the Dirac fermion.
Compute the Noether's currents J_L^μ and J_R^μ associated to these symmetries.
Consider the combinations

$$J_V^\mu = J_R^\mu + J_L^\mu, \qquad\qquad J_A^\mu = J_R^\mu - J_L^\mu.$$

Show that these are the Noether's currents associated to the following symmetry transformation acting on the Dirac fermion:

$$\begin{aligned} U(1)_V: &\ \ \psi \longrightarrow e^{i\alpha}\psi, \\ U(1)_A: &\ \ \psi \longrightarrow e^{i\beta\gamma^5}\psi, \end{aligned}$$

where $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$.
What changes if one adds a mass term for the Dirac fermion: $m\bar{\psi}\psi$?

Exercise 2: Angular momentum in the Dirac theory

Starting from the Dirac Lagrangian (written in its hermitian form) compute the Noether's current $M_{\mu\nu}^{\rho}$ associated to Lorentz invariance.
Introduce the angular momentum operator

$$J^k \equiv \frac{1}{2}\epsilon^{ijk}\int d^3x\ M_{ij}^0.$$

Show that it can be written as

$$J^k = \int d^3x\ \psi^\dagger(t,\vec{x})(L_k + \Sigma_k/2)\psi(t,\vec{x}),$$

where

$$\begin{aligned} L^k &= [\vec{x}\wedge(-i\vec{\nabla})]^k, \\ \Sigma^k &= \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}, \end{aligned}$$

are respectively the orbital and spin parts.
Defining

$$\vec{S} = \int d^3x\ \psi^\dagger(t,\vec{x})\frac{\vec{\Sigma}}{2}\psi(t,\vec{x}),$$

compute the eigenvalue of the operator $\vec{S}^2$ on a generic one-particle state in position space, $\psi_\alpha^\dagger(t,\vec{x})|0\rangle \equiv |x,\alpha\rangle$, and show this way that $\psi^\dagger$ creates states with spin one half.

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References & Exercises

- "A Modern Introduction to Quantum Field Theory", Maggiore:
  Paragraphs: 1.2, 2.1-2.7, 3.1-3.4, 4.1-4.2.2
- "An Introduction to Quantum Field Theory", Peskin & Schroeder:
  Pages: 13-26, 35-62

- "An Introduction to Quantum Field Theory", Peskin & Schroeder:
  Problems **2.1**, **3.4**, **3.5**
- "A Modern Introduction to Quantum Field Theory", Maggiore:
  Problems **2.1**-**2.5**, **3.1**, **3.5**, **3.6** (solutions at the end of the book)