Quantum Field Theory
Set 14

Exercise 1: U(1) symmetry and chiral symmetry

Given the Lagrangian density of a massless Dirac fermion \( \psi \equiv \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \):

\[
L = i \bar{\psi} \gamma^\mu \partial_\mu \psi,
\]

verify that it is invariant under a symmetry \( U(1)_L \times U(1)_R \) where each \( U(1) \) acts independently on the left or right component of the Dirac fermion.

Compute the Noether’s currents \( J^\mu_L \) and \( J^\mu_R \) associated to these symmetries.

Consider the combinations

\[
J^\mu_V = J^\mu_R + J^\mu_L, \quad J^\mu_A = J^\mu_R - J^\mu_L.
\]

Show that these are the Noether’s currents associated to the following symmetry transformation acting on the Dirac fermion:

\[
U(1)_V : \psi \rightarrow e^{i\alpha} \psi, \\
U(1)_A : \psi \rightarrow e^{i\beta \gamma^5} \psi,
\]

where \( \gamma^5 \equiv i\gamma^0 \gamma^1 \gamma^2 \gamma^3 \).

What changes if one adds a mass term for the Dirac fermion: \( m \bar{\psi} \psi \)?

Exercise 2: Angular momentum in the Dirac theory

Starting from the Dirac Lagrangian (written in its hermitian form) compute the Noether’s current \( M^\mu_\nu \) associated to Lorentz invariance.

Introduce the angular momentum operator

\[
J^k \equiv \frac{1}{2} \epsilon^{ijk} \int d^3x \, M^0_{ij},
\]

Show that it can be written as

\[
J^k = \int d^3x \, \psi^\dagger(t, \vec{x})(L^k + \Sigma_k/2)\psi(t, \vec{x}),
\]

where

\[
L^k = [\vec{x} \wedge (-i \vec{\nabla})]^k, \\
\Sigma^k = \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix},
\]

are respectively the orbital and spin parts.

Defining

\[
\vec{S} = \int d^3x \, \psi^\dagger(t, \vec{x}) \frac{\Sigma}{2} \psi(t, \vec{x}),
\]

compute the eigenvalue of the operator \( \vec{S}^2 \) on a generic one-particle state in position space, \( \psi^\dagger_\alpha(t, \vec{x})|0\rangle \equiv |x, \alpha\rangle \), and show this way that \( \psi^\dagger \) creates states with spin one half.
Quantum Field Theory
References & Exercises

- “A Modern Introduction to Quantum Field Theory”, Maggiore:
  Paragraphs: 1.2, 2.1-2.7, 3.1-3.4, 4.1-4.2.2
- “An Introduction to Quantum Field Theory”, Peskin & Schroeder:
  Pages: 13-26, 35-62
- “An Introduction to Quantum Field Theory”, Peskin & Schroeder:
  Problems 2.1, 3.4, 3.5
- “A Modern Introduction to Quantum Field Theory”, Maggiore:
  Problems 2.1-2.5, 3.1, 3.5, 3.6 (solutions at the end of the book)