Exercise 1: Coherent states

A state like $|\chi\rangle \equiv a^\dagger(\vec{k}_1)a^\dagger(\vec{k}_2)\ldots a^\dagger(\vec{k}_n)|0\rangle$ describes a definite number of particles with spatial momenta $\vec{k}_1, \vec{k}_2, \ldots, \vec{k}_n$. Instead, a coherent state in quantum mechanics is an eigenvector of the annihilation operators, $a(\vec{q})|\psi\rangle = \alpha(\vec{q})|\psi\rangle$, where the eigenvalue $\alpha$ is a complex function of the momenta. Obviously $|\chi\rangle$ defined above cannot be a coherent states, since it has a definite number of particles.

Using the normalization for the ladder operators $[a(\vec{q}), a^\dagger(\vec{p})] = \delta^3(\vec{k} - \vec{q})$, solve the following points:

- Find the general form of a coherent state, disregarding for the moment the normalization factor. Hint: start from the ansatz $|\psi\rangle = \sum_{n=0}^{\infty} c_n \left( \int dk z(\vec{k}) a^\dagger(\vec{k}) \right)^n |0\rangle$ with $z(\vec{k})$ a generic complex functions and the coefficients $c_n$ to be fixed. How can the result be written in a compact way? Hint: Use the formula $[A, B^n] = [A, B]B^{n-1} + B[A, B]B^{n-2} + \ldots + B^{n-1}[A, B]$.

- Put the right normalization factor in front of $|\psi\rangle$, so that $<\psi|\psi> = 1$ (Assume that the vacuum is normalized to 1, $\langle 0|0\rangle = 1$).

- Compute the expectation value $<\psi|\phi(x)|\psi>$ of the Klein-Gordon field $\phi(x) = \int \frac{d^3k}{(2\pi)^3/2} \sqrt{k^0} \left[ a(\vec{k})e^{-ikx} + a^\dagger(\vec{k})e^{ikx} \right]$. Is the result different from the case of a state with definite particle number?