Advanced Quantum Field Theory

Exercise 11

Consider a field theory at finite temperature based on the following Lagrangian:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \tag{1}$$

The Free Energy of the system can be computed as a perturbative series around the configuration $\lambda = 0$:

$$\begin{aligned} Z[\beta] &= \int \mathcal{D}\phi e^{-S_0}e^{-S_I} = \int \mathcal{D}\phi e^{-S_0}\sum_{l=0}^{\infty}\frac{1}{l!}(-S_I)^l \\ &= \int \mathcal{D}\phi e^{-S_0}\sum_{l=0}^{\infty}\frac{1}{l!}\langle(-S_I)^l\rangle_0 \\ F &= F_0 + F_I = -\frac{1}{\beta V}\langle -S_I\rangle + \ldots \end{aligned} \tag{2}$$

where $S_0 + S_I$ is the euclidean action:

$$S_0 + S_I = \int d^4x\left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4\right) \tag{3}$$

In order to determine the first term in (2) compute first the free propagator expanding the field ϕ in Fourier modes (recall that the time is compactified on a circle)

$$\phi(\vec{x},\tau) = \sum_{k=-\infty}^{+\infty}\sqrt{\beta}\phi_k(\vec{x})e^{iw_k\tau}\,, \qquad \text{with} \quad \phi_{-k} = \phi_k^*\,, \quad w_k = 2\pi kT\,. \tag{4}$$

The propagator is defined as

$$\langle\phi(x_1)\phi(x_2)\rangle_0 \tag{5}$$

Finally, using the Wick theorem, compute:

$$F_I = \frac{1}{\beta V}\int d^4x\langle\frac{\lambda}{4!}\phi(x)\phi(x)\phi(x)\phi(x)\phi(x)\rangle_0 + \text{perm.} \tag{6}$$

Notice that the above term contains a $T-$dependent divergent piece.