**Series 9**

The partition function of the Ising model in 1 dimension and periodic boundary conditions ($\sigma_{N+1} = \sigma_1$) is:

$$Z = \sum_{\{\sigma\}} \prod_i^N \exp\left(K\sigma_i\sigma_{i+1} + \frac{1}{2}H(\sigma_i + \sigma_{i+1})\right)$$

with $K = \beta J$ et $H = \beta h$.

The aim of the renormalization group is to reduce the number of degrees of freedom with no change the form of the partition function (up to constant factors). Here we will apply the renormalization procedure of one spin over two. We perform the sum over $\sigma_2, \sigma_4, \ldots$ in the partition function in such a way that the new partition function contains terms like $C\exp\left(K'\sigma_i\sigma_{i+2} + \frac{1}{2}H'(\sigma_i + \sigma_{i+2})\right)$ with odd $i$ and $C$ is a constant factor.

a) Identify the terms in $\sigma_i$ and $\sigma_{i+2}$ of the new partition function with the sum over $\sigma_{i+1}$ in the initial function (consider the case $i = 1$).

b) Calculate $C, K'$ and $H'$ as a function of $K$ and $H$ evaluating the equations of point a) at the possible values of $\sigma_1$ and $\sigma_3$.

c) Verify that the points $(K^* = 0, H)$ (paramagnet) and $(K^* = \infty, H^* = 0)$ (ferromagnet) are fixed points of the renormalization transformation.

d) For small $H$, show that the transformation is:

$$\begin{cases} \tanh(K') = \tanh^2(K) \\ \\ H' = H\,(1 + \tanh(2K)) \end{cases}$$

e) Posing $v = \tanh(K)$, linearize the transformation around $(v^* = 1, H^* = 0)$ and deduce the stability of the equilibrium point. Do the same around $v^* = 0$.

Remember that:

$$\cosh(2x) = \sinh^2(x) + \cosh^2(x)$$

$$\cosh^2(x) - \sinh^2(x) = 1$$

$$\tanh(2x) = \frac{2\tanh(x)}{1 + \tanh^2(x)}$$

$$\ln\left(\frac{1+x}{1-x}\right) = 2x + \mathscr{O}(x^2)$$

Reference: *Introduction to renormalization group methods in physics*, Creswick, Farach et Poole, 1992 (play attention to the typos in one of the equations...).