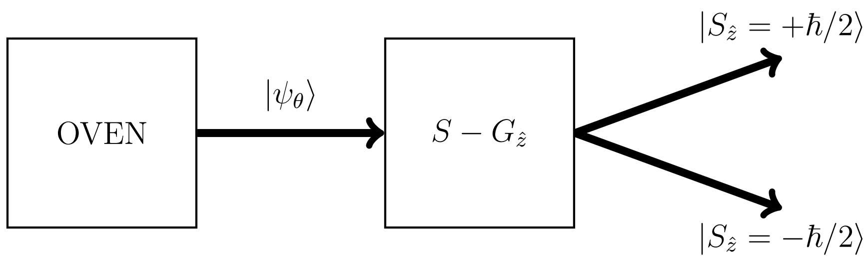
1. The initial state of an electron leaving the oven is

$$|\psi_\theta\rangle = \cos(\theta)|S_{\hat{z}} = +\hbar/2\rangle + \sin(\theta)|S_{\hat{z}} = -\hbar/2\rangle \, ,$$

with θ distributed according the probability density function

$$\rho(\theta) = \begin{cases} \dfrac{1}{2\pi} \text{ if } \theta \in [0, 2\pi] \\ 0 \text{ otherwise.} \end{cases}$$



Hence, the probability that the first Stern and Gerlach experiment measures $S_{\hat{z}} = +\hbar/2$, for a fixed θ, is given by

$$\mathcal{P}\left(S_{\hat{z}} = +\hbar/2; \theta\right) = |\langle S_{\hat{z}} = +\hbar/2|\psi_\theta\rangle|^2 = \cos^2(\theta) \, ,$$

while the probability that the first Stern and Gerlach experiment measures $S_{\hat{z}} = -\hbar/2$, for a fixed θ, is given by

$$\mathcal{P}\left(S_{\hat{z}} = +\hbar/2; \theta\right) = |\langle S_{\hat{z}} = +\hbar/2|\psi_\theta\rangle|^2 = \sin^2(\theta) \, .$$

Averaging over $\rho(\theta)$ yields how the electrons distribute among the two beams:

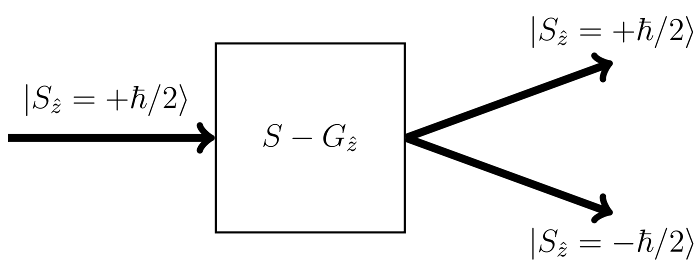
$$\begin{aligned}
\mathcal{P}\left(S_{\hat{z}} = +\hbar/2\right) &= \int_{-\infty}^{\infty} \mathcal{P}\left(S_{\hat{z}} = +\hbar/2; \theta\right) \rho(\theta) d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} \cos^2(\theta) d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} \frac{1 + \cos(2\theta)}{2} d\theta \\
&= \frac{1}{2\pi} \left[\frac{x}{2} + \frac{\sin(2\theta)}{4} \right]_0^{2\pi} \\
&= \frac{1}{2} \, ;
\end{aligned}$$

$$
\begin{aligned}
\mathcal{P}\left(S_{\hat{z}} = -\hbar/2\right) &= \int_{-\infty}^{\infty} \mathcal{P}\left(S_{\hat{z}} = -\hbar/2; \theta\right) \rho(\theta) d\theta \\
&= \frac{1}{2\pi} \int_{0}^{2\pi} \sin^2(\theta) d\theta \\
&= \frac{1}{2\pi} \int_{0}^{2\pi} \frac{1 - \cos(2\theta)}{2} d\theta \\
&= \frac{1}{2\pi} \left[\frac{x}{2} - \frac{\sin(2\theta)}{4} \right]_0^{2\pi} \\
&= \frac{1}{2} \, .
\end{aligned}
$$

2. Now, the initial state is $|S_{\hat{z}} = +\hbar/2\rangle$, therefore the probabilities that the second Stern and Gerlach experiment measures $S_{\hat{z}} = -\hbar/2$ and $S_{\hat{z}} = -\hbar/2$ are

$$
\begin{aligned}
\mathcal{P}\left(S_{\hat{z}} = +\hbar/2\right) &= |\langle S_{\hat{z}} = +\hbar/2 | S_{\hat{z}} = +\hbar/2\rangle|^2 = 1 \, , \\
\mathcal{P}\left(S_{\hat{z}} = -\hbar/2\right) &= |\langle S_{\hat{z}} = -\hbar/2 | S_{\hat{z}} = +\hbar/2\rangle|^2 = 0 \, ,
\end{aligned}
$$

respectively.

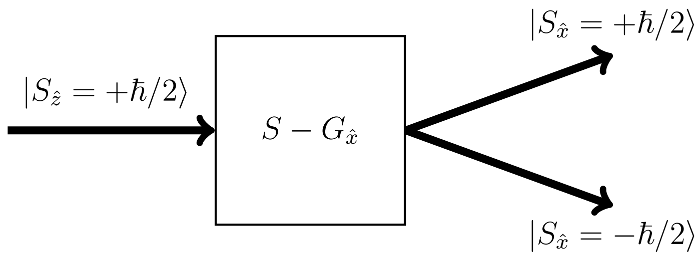


3. The first step to answer to this point is to compute $|S_{\hat{x}} = +\hbar/2\rangle$ and $|S_{\hat{x}} = -\hbar/2\rangle$. These two kets are defined by the equations

$$
S_{\hat{x}} |S_{\hat{x}} = \pm\hbar/2\rangle = \pm\hbar/2 |S_{\hat{x}} = \pm\hbar/2\rangle \, ,
$$

which give

$$
\begin{aligned}
|S_{\hat{x}} = +\hbar/2\rangle &= \frac{|S_{\hat{z}} = +\hbar/2\rangle + |S_{\hat{z}} = -\hbar/2\rangle}{\sqrt{2}} \, , \\
|S_{\hat{x}} = -\hbar/2\rangle &= \frac{|S_{\hat{z}} = +\hbar/2\rangle - |S_{\hat{z}} = -\hbar/2\rangle}{\sqrt{2}} \, .
\end{aligned}
$$



Finally, the probabilities that the third Stern and Gerlach experiment measures $S_{\hat{x}} = -\hbar/2$ and $S_{\hat{x}} = -\hbar/2$ are

$$\begin{aligned}
\mathcal{P}\left(S_{\hat{x}} = +\hbar/2\right) &= \left|\langle S_{\hat{x}} = +\hbar/2 | S_{\hat{z}} = +\hbar/2\rangle\right|^2 \\
&= \left| \frac{1}{\sqrt{2}} \Big(\langle S_{\hat{z}} = +\hbar/2| + \langle S_{\hat{z}} = -\hbar/2| \Big) |S_{\hat{z}} = +\hbar/2\rangle \right|^2 \\
&= \left| \frac{1}{\sqrt{2}} \Big(\langle S_{\hat{z}} = +\hbar/2 | S_{\hat{z}} = +\hbar/2\rangle + \langle S_{\hat{z}} = -\hbar/2 | S_{\hat{z}} = +\hbar/2\rangle \Big) \right|^2 \\
&= \frac{1}{2}\ ; \\
\mathcal{P}\left(S_{\hat{x}} = -\hbar/2\right) &= \left|\langle S_{\hat{x}} = -\hbar/2 | S_{\hat{z}} = +\hbar/2\rangle\right|^2 \\
&= \left| \frac{1}{\sqrt{2}} \Big(\langle S_{\hat{z}} = +\hbar/2| - \langle S_{\hat{z}} = -\hbar/2| \Big) |S_{\hat{z}} = +\hbar/2\rangle \right|^2 \\
&= \left| \frac{1}{\sqrt{2}} \Big(\langle S_{\hat{z}} = +\hbar/2 | S_{\hat{z}} = +\hbar/2\rangle - \langle S_{\hat{z}} = -\hbar/2 | S_{\hat{z}} = +\hbar/2\rangle \Big) \right|^2 \\
&= \frac{1}{2}\ .
\end{aligned}$$

4. In this case, the initial state is $|S_{\hat{x}} = +\hbar/2\rangle$, therefore, the probabilities that the fourth Stern and Gerlach experiment measures $S_{\hat{z}} = -\hbar/2$ and

$S_{\hat{z}} = -\hbar/2$ are

$$\begin{aligned}
\mathcal{P}\left(S_{\hat{z}} = +\hbar/2\right) &= \left|\langle S_{\hat{z}} = +\hbar/2 | S_{\hat{x}} = +\hbar/2\rangle\right|^2 \\
&= \left|\frac{1}{\sqrt{2}}\langle S_{\hat{z}} = +\hbar/2|\Big(|S_{\hat{z}} = +\hbar/2\rangle + |S_{\hat{z}} = -\hbar/2\rangle\Big)\right|^2 \\
&= \left|\frac{1}{\sqrt{2}}\Big(\langle S_{\hat{z}} = +\hbar/2|S_{\hat{z}} = +\hbar/2\rangle + \langle S_{\hat{z}} = +\hbar/2|S_{\hat{z}} = -\hbar/2\rangle\Big)\right|^2 \\
&= \frac{1}{2}\ ,
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{P}\left(S_{\hat{z}} = -\hbar/2\right) &= \left|\langle S_{\hat{z}} = -\hbar/2 | S_{\hat{x}} = +\hbar/2\rangle\right|^2 \\
&= \left|\frac{1}{\sqrt{2}}\langle S_{\hat{z}} = -\hbar/2|\Big(|S_{\hat{z}} = +\hbar/2\rangle - |S_{\hat{z}} = -\hbar/2\rangle\Big)\right|^2 \\
&= \left|\frac{1}{\sqrt{2}}\Big(\langle S_{\hat{z}} = -\hbar/2|S_{\hat{z}} = +\hbar/2\rangle - \langle S_{\hat{z}} = -\hbar/2|S_{\hat{z}} = -\hbar/2\rangle\Big)\right|^2 \\
&= \frac{1}{2}\ .
\end{aligned}$$