Notes on Lie Groups

1 Lie Group/Lie Algebra relation

- Abstract mathematical result (Lie's theorems I, II and III): the tangent space T_g at each element g in a lie group G defines a Lie Algebra with dimensionality equal to the dimensionality of G.
- We shall not prove this result by working with the abstract group. We shall instead derive it by taking for granted the existence of matrix representations for any Lie Group. For our physics oriented purposes this approach is more than satisfactory. It also simplifies things quite a bit, since the existence of matrix representations is itself a non trivial mathematical property of abstract Lie Groups (Ado's theorem), which we assume here without proof.

Let us take a Lie group G, parametrized by $\alpha^1, \dots, \alpha^n \equiv \{\alpha\} \in \mathbb{R}^n$. We choose $\alpha = 0$ to correspond with the identity: $g(0) = e$. Let us now consider a representation

$$g(\alpha) \mapsto D(g(\alpha)) \equiv D(\alpha) \tag{1}$$

The tangent space describes a "linearization" of a manifold in the neighbourhood of a point. In the matrix representation, the tangent space at a point α_0 is therefore associated to the expansion of $D(\alpha)$ around α_0 up to linear order in $\alpha - \alpha_0$. Around the origin $\alpha_0 = 0$ we have

$$D(\alpha) = \mathbf{1} + i\alpha^k X_k + O(\alpha^2) \tag{2}$$

where

$$iX_k \equiv \frac{\partial}{\partial \alpha^k} D(\alpha)|_{\alpha=0} \tag{3}$$

are a set of n matrices. The normalization factor i is chosen for later use (by this convention X is hermitian when $D(\alpha)$ in unitary).

The matrices X_j are called the generators of the group. We are now going to show, in two steps, that the X_k form a Lie algebra.

Step 1 We first show that X_k describe the expansion of $D(\alpha)$ around any point, not just around the identity. Consider the group product

$$D(\alpha)D(\beta) = D(p(\alpha, \beta)) \tag{4}$$

and differentiate it with respect to β

$$D(\alpha)\frac{\partial D(\beta)}{\partial \beta^j} = \frac{\partial D(p)}{\partial p^k}\frac{\partial p^k}{\partial \beta^j}\,. \tag{5}$$

Setting $\beta = 0$ and noting that $p(\alpha, 0) = \alpha$, the above equation becomes

$$D(\alpha)iX_j = \frac{\partial D(\alpha)}{\partial \alpha^k} u_j^k(\alpha) \tag{6}$$

By the existence and differentiability of the inverse of $p(\alpha,\beta)$ (which follows from the definition of Lie group) it follows that $u^k_j(\alpha) = \partial p^k(\alpha,\beta)/\partial\beta^j|_{\beta=0}$ is an invertible matrix. Indeed, since $p(0,\beta)^k = \beta^k$, at $\alpha = 0$ we have $u^k_j(0) = \partial\beta^k/\partial\beta^j = \delta^k_j$. Let us call $v^j_j(\alpha)$ the inverse: $u^k_j(\alpha)v^j_i(\alpha) = \delta^k_i$. Multiplying both sides of 6 by $D(\alpha)^{-1}$ and v^j_i we have

$$iT_i(\alpha) \equiv D(\alpha)^{-1}\frac{\partial D(\alpha)}{\partial\alpha^i} = iX_j v^j_i(\alpha) \tag{7}$$

Notice that $T_i(0) = X_i$ This equation shows that the matrices T_k describing the differential of D at an arbitrary point α, are linear combinations of X_k.

$$D(\alpha) = D(\alpha_0)\left[1 + i(\alpha-\alpha_0)^k v^j_k(\alpha_0)X_j + \dots\right] \tag{8}$$

Mathematically this result can be phrased by saying that the tangent space at any point is isomorphic to the tangent space at the identity.

Step 2 Let us now compute the curl of both sides of eq. 7 to get a new identity. That is we compute the quantity

$$i(\partial_j T_k(\alpha) - \partial_k T_j(\alpha)) \tag{9}$$

using the two ways of writing T given in eq. 7. To simplify notation we use from now on $\partial_k \equiv \partial/\partial\alpha^k$.

a) Compute first eq.9 using $iT_i(\alpha) = D(\alpha)^{-1}\partial_i D(\alpha)$

$$\partial_j(D^{-1}\partial_k D) - \partial_k(D^{-1}\partial_j D) = \partial_j D^{-1}\partial_k D + D^{-1}\partial_j\partial_k D - \partial_k D^{-1}\partial_j D - D^{-1}\partial_j\partial_k D = \tag{10}$$

$$= \partial_j D^{-1}\partial_k D - \partial_k D^{-1}\partial_j D \tag{11}$$

The result can be written in a more illuminating form by using the identity

$$0 = \partial_k(1) = \partial_k(D^{-1}D) = \partial_k D^{-1}D + D^{-1}\partial_k D \tag{12}$$

which by can also be written as

$$\partial_k D^{=1} = -D^{-1}\partial_k D D^{-1} \tag{13}$$

Therefore we can write eq. 11 as

$$\partial_j D^{-1}\partial_k D - \partial_k D^{-1}\partial_j D = -(D^{-1}\partial_j D)(D^{-1}\partial_k D) + (D^{-1}\partial_k D)(D^{-1}\partial_j D) \tag{14}$$

$$= [T_j(\alpha), T_k(\alpha)] \tag{15}$$

$$= [X_m, X_n]v^m_j(\alpha)v^n_k(\alpha) \tag{16}$$

b) Compute eq.9 using $iT_i(\alpha) = iX_j v^j_i(\alpha)$

$$iX_m\left(\partial_j v^m_k(\alpha) - \partial_k v^m_j(\alpha)\right) \tag{17}$$

Equating eq.16 and eq. 17

$$[X_m, X_n]v^m_j(\alpha)v^n_k(\alpha) = iX_m\left(\partial_j v^m_k(\alpha) - \partial_k v^m_j(\alpha)\right) \tag{18}$$

and multiplying by $u^j_r(\alpha)u^k_s(\alpha)$ we find at the end

$$[X_r, X_s] = iX_\ell \left(\partial_j v^\ell_k(\alpha) - \partial_k v^\ell_j(\alpha)\right) u^j_r(\alpha)u^k_s(\alpha) = iX_\ell f^\ell_{rs} \tag{19}$$

showing that

1. $$f^\ell_{rs} = \left(\partial_j v^\ell_k(\alpha) - \partial_k v^\ell_j(\alpha)\right) u^j_r(\alpha)u^k_s(\alpha) = \left(\partial_j v^\ell_k(0) - \partial_k v^\ell_j(0)\right) u^j_r(0)u^k_s(0) \tag{20}$$
   do not depend on α. This is a quite non-trivial constraint on the product function $p(\alpha, \beta)$ from which u^k_m is derived.

2. X_i form a Lie algebra with structure constants $= f^\ell_{rs}$

3. f^ℓ_{rs} only depend on the abstract group product $p(\alpha, \beta)$ and thus are the same for all representations of the group in consideration!

We can thus associate one (and only one) Lie Algebra to any given Lie group . The converse is not true: there is in general more than one group associated to a given Lie Algebra.

More picky mathematical detail

In our discussion we used at a certain point the invertibility of the differential $u^k_j = \partial p^k(\alpha, \beta)/\partial\beta^j|_{\beta=0}$. This result holds true provided $p^k(\alpha, \beta)$, viewed as a function of β $p^k(\alpha, \beta) \equiv f^k_\alpha(\beta)$, has a differentiable inverse (i.e. $(u^{-1})^j_k = \partial\beta^j/\partial p^k$) . This property indeed follows from our assumption of differentiability of the group product. Defining $g^{-1}(\alpha) = g(r(\alpha))$ we have

$$g(\beta) = g(r(\alpha))g(\alpha)g(\beta) = g\Big(p(r(\alpha), p(\alpha, \beta))\Big) \tag{21}$$

that is

$$\beta = p(r(\alpha), p(\alpha, \beta)) \qquad \longrightarrow \qquad f^{-1}_\alpha(y) = p(r(\alpha), y) \tag{22}$$

which shows explicitly that the inverse function f^{-1}_α is a composition of differentiable functions, and is thus differentiable.