Lecture #14

Gravitational waves.

We should expect to have gravitational waves - perturbations of the metric - to exist in GR, in complete analogy with electrodynamics

Weak gravitational waves:

$g_{\mu\nu} = \eta_{\mu\nu} + \delta g_{\mu\nu}$, $\delta g_{\mu\nu} << 1$.

wave equation in empty space:

$R_{\mu\nu} = 0$, where $R_{\mu\nu}$ is expressed via $\delta g_{\mu\nu}$.

Linear analysis: find $R_{\mu\nu}$ as a function of $\delta g_{\mu\nu}$, and keep only linear terms. We will get linear differential equations $\Rightarrow$ we have superposition principle and may work in Fourier space.

choose: $\delta g_{\mu\nu} = h_{\mu\nu} \exp(i k_\mu x^\mu)$

wave propagating in direction $\vec{k}$

Let us first consider this problem for electrodynamics:

$$\partial_\mu F^{\mu\nu} = 0 \qquad \text{for } A_\mu \to A_\mu e^{ik_\nu x^\nu}$$

we get $(k^2 g_{\mu\nu} - k_\mu k_\nu) A^\nu = 0$

Let us choose wave propagation in direction z : $k_3 = k \neq 0$, $k_1 = k_2 = 0$; $k_0 = \omega$

Then for A^1, A^2 equations are:

$$(\omega^2 - \vec{k}^2) A^1 = 0 \ ; \ (\omega^2 - \vec{k}^2) A^2 = 0 \qquad (1)$$

and the pair of equations for A^0, A^3 gives

$$\omega A_3 - k A_0 = 0 \qquad (2)$$

Interesting (and very important) fact: number of variables is 4 (A_μ) but number of equations is 3 – one of variables is arbitrary. Reason: gauge invariance: A_μ and $A_\mu - \partial_\mu \alpha$ are equivalent. in Fourier space we have: $\alpha \sim e^{i\omega t - ikz}$, A_1 and A_2 do not change, $A_0 \to A_0 - i\omega\alpha$; $A_3 \to A_3 - ik\alpha$

Let us use this freedom and chose

$A_0 = 0$ (can always be made if

initially $A_0 \neq 0$: make gauge transformation

with $\alpha = A_0/(i\omega)$). Then, from eq. (2),

$A_3 = 0$.

Solutions for A_1, A_2 (which are gauge-invariant) are plane waves with $\omega = \pm k$:

$$\begin{cases} A_1 = c_1 \exp[-ik(z \pm t)] \\ A_2 = c_2 \exp[-ik(z \pm t)] \\ A_0 = A_3 = 0 \end{cases}$$

they propagate with speed of light, and characterized by two amplitudes c_1 and c_2 - corresponding to 2 possible polarizations.

(massless spin 2 particle - photon, in quantum field theory)

For gravitational field:

$$R_{\mu\nu} = \frac{1}{2}\left(k^2 h_{\mu\nu} - k_\lambda k_\mu h^\lambda{}_\nu - k_\lambda k_\nu h^\lambda{}_\mu + \right.$$
$$\left. + k_\mu k_\nu h^\lambda{}_\lambda\right)$$

$$R = \frac{1}{2}\left(k^2 h^\lambda{}_\lambda - k_\lambda k_\mu h^{\lambda\mu} \cdot 2 + k^2 h^\lambda{}_\lambda\right);$$

$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \approx$ insert $R_{\mu\nu}$ and R from above, replace $g_{\mu\nu}$ by $\eta_{\mu\nu}$

All 10 equations are given on the next page.

We will get then 10 equations for 10 amplitudes $h_{\mu\nu}$, listed below for $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ – E. tensor

00 : $(h_{11} + h_{22}) k^2 = 0$ ✓

03 : $(h_{11} + h_{22}) \omega k = 0$ ✓

02 : $(-h_{20} k + h_{23} \omega) k = 0$ ✓

01 : $(-h_{10} k + h_{13} \omega) k = 0$ ✓

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33 : $(h_{11} + h_{22}) \omega^2 = 0$ ✓

32 : $(-h_{20} k + h_{23} \omega) \omega = 0$ ✓

31 : $(-h_{10} k + h_{13} \omega) \omega = 0$ ✓

00; 03; 33 – same

02; 32 – the same

01; 31 – the same

visibly, 6 different equations for 10 unknowns

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22 : $h_{00} k^2 + \omega(-2h_{30} k + h_{33} \omega) + h_{11}(\omega^2 - k^2) = 0$

21 : $h_{12}(\omega^2 - k^2) = 0$

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11 : $h_{00} k^2 + \omega(-2h_{30} k + h_{33} \omega) + h_{22}(\omega^2 - k^2) = 0$

independent 6 equations are given on the next page.

3 equations $(00,\ 03,\ 33)$: $h_{11} + h_{22} = 0$

2 equations $(02,\ 32)$: $-h_{20}k + h_{23}\omega = 0$

2 eq $(01,\ 31)$: $-h_{10}k + h_{13}\omega = 0$

1 eq (21) : $h_{12}(\omega^2 - k^2) = 0$

1 eq $[22 - 11]$: $(h_{11} - h_{22})(\omega^2 - k^2) = 0$

1 eq $[22 + 11]$: $h_{00}k^2 + \omega(-2h_{30}k + h_{33}\omega) = 0$

4 variables cannot be determined.

Reason: the same as in Electrodynamics.

We have a freedom to make arbitrary coordinate transformations:

coordinate transformations:

$x^\mu \to x'^\mu = x^\mu + \xi^\mu$, where ξ^μ is small $\Rightarrow$

$$g'^{\mu\nu} = \frac{\partial x'^\mu}{\partial x^\alpha}\frac{\partial x'^\nu}{\partial x^\beta} g^{\alpha\beta} \Rightarrow$$ in linear order $h_{\mu\nu} \to h'_{\mu\nu} = h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu$

Coordinate transformations:

$h_{00} \to h_{00} - 2i\omega\xi_0$

$h_{01} \to h_{01} - i\omega\xi_1$

$\underline{h_{02}} \to h_{02} - i\omega\xi_2$

$h_{03} \to h_{03} - i\omega\xi_3 - ik\xi_0$

$h_{13} \to h_{13} - ik\xi_1$

$\underline{h_{23}} \to h_{23} - ik\xi_2$

$h_{33} \to h_{33} - 2ik\xi_3$

without change:

$h_{11} \to h_{11}$

$h_{22} \to h_{22}$

$h_{12} \to h_{12}$

So, h and h_{12} are physical - do not change if we change coord. system

Let us use this freedom to put some components of $h_{\mu\nu}$ to zero:

ξ_0 : put h_{00} to zero, $\xi_0 = (h_{00})^{\text{initial}}/(2i\omega)$

ξ_3 : put h_{33} to zero, $\xi_3 = (h_{33})^{\text{initial}}/(2ik)$

ξ_2 : put h_{02} to zero, $\xi_2 = (h_{02})^{\text{initial}}/(i\omega)$

ξ_1 : put h_{10} to zero, $\xi_1 = (h_{01})^{\text{initial}}/(i\omega)$

After this is made, our 6 equations are:

$$h_{11} + h_{22} = 0 \qquad \Longrightarrow \qquad h_{22} = -h_{11} \equiv -h$$

$$h_{23} = 0$$

$$h_{13} = 0 \qquad\qquad h_{00} = h_{33} = h_{01} = h_{02} = h_{03}$$

$$h_{12}(\omega^2 - k^2) = 0 \qquad\qquad = h_{13} = h_{23} = 0$$

$$(h_{11} - h_{22})(\omega^2 - k^2) = 0 \qquad \Longrightarrow \qquad h(\omega^2 - k^2) = 0$$

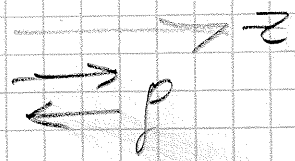
$$h_{30} = 0$$

<u>Conclusion</u>: for gravitational wave $\parallel z$ we have following physical components:

$$h_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h & h_{12} & 0 \\ 0 & h_{12} & -h & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{i\omega t - ikz}$$

two tensorial „polarizations"

in quantum theory: spin 2, 2 polarizations:



„graviton" – particle, corresponding to gravitational wave

Emission of gravitational waves

* we are here $\xrightarrow{\quad\quad}$ z  matter distribution, moving particles

Relevant equations:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G \, T_{\mu\nu}$$

For physical amplitudes, in linear approximation:

$$\Box h_{12} = 8\pi G \, T_{12} \qquad \Box = \omega^2 - k^2 \text{ in momentum space}$$

$$\Box h = 8\pi G \, \frac{1}{2}\left[T_{22} - T_{11}\right]$$

we have already solved these type of equations in lectures on Electrodynamics

Solution: via retarded Green's functions

$$G(x, t; x', t'):$$

$$h_{12}(x,t) = \int 8\pi G \, T_{12}(x', t') \, G(x, t; x', t') \, d^3x' dt'$$

$$= 2G \int T_{12}(x', \underbrace{t - |x - x'|}_{\text{retarded time!}}) \frac{1}{|x - x'|} \, d^3x' \approx$$

$$\approx \frac{2G}{R} \int d^3x' \, T_{12}(x', t - R) \qquad \text{valid if } \underline{v} \ll c$$

velocities of particles

Computation of integrals

$\int d^3x\, T_{\mu\nu}$: similar what we did with currents in electrodynamics

EM conservation:

$$\frac{\partial T_{\alpha\gamma}}{\partial x^\gamma} - \frac{\partial T_{\alpha 0}}{\partial x^0} = 0 \qquad \frac{\partial T_{0\gamma}}{\partial x^\gamma} - \frac{\partial T_{00}}{\partial x^0} = 0$$

First eq: multiply by x^β and integrate over space:

$$\frac{\partial}{\partial x^0}\int T_{\alpha 0}\, x^\beta\, dV = \int \frac{\partial T_{\alpha\gamma}}{\partial x^\gamma}\, x^\beta\, d^3x =$$

$$= \int \Bigg[\underbrace{\frac{\partial\left[T_{\alpha\gamma}\, x^\beta\right]}{\partial x^\gamma}}_{\text{vanished, Gauss theorem}} - T_{\alpha\beta} \Bigg] dV \Rightarrow$$

$$\int T_{\alpha\beta}\, dV = -\frac{1}{2}\frac{\partial}{\partial x^0}\int \left(T_{\alpha 0}\, x^\beta + T_{\beta 0}\, x^\alpha\right) dV$$

Second eq: multiply by $x^\alpha x^\beta$, and integrate $\Rightarrow$

$$\frac{\partial}{\partial x^0}\int T_{00}\, x^\alpha x^\beta\, dV = -\int \left(T_{\alpha 0}\, x^\beta + T_{\beta 0}\, x^\alpha\right) dV$$

So,

$$\int T_{\alpha\beta}\, dV = \frac{1}{2}\frac{\partial^2}{(\partial x^0)^2}\int \underset{\rho(x)\ -\ \text{mass distribution}}{T_{00}}\, x^\alpha x^\beta\, dV$$

Quadropole moment of mass distribution:

$$D^{ij} = \int \rho(x)\left[3x^i x^j - r^2 \delta^{ij}\right] dV \Rightarrow$$

$$h_{12} \approx \frac{2G}{3R} \ddot{D}_{12} \quad ; \quad h = \frac{2G}{3R} \cdot \frac{1}{2}\left(\ddot{D}_{22} - \ddot{D}_{11}\right)$$

Energy loss, order of magnitude estimate:

Energy density of gravitational wave

Since h is analogue of vector potential, energy density of gravitational wave must contain $(\dot{h})^2$ or $(\dot{h}_{12})^2$ (GeV^2)

energy density: GeV^4 $\Rightarrow$

$$\varepsilon \propto G^{-1}(\dot{h})^2$$

Lost of energy $\sim \frac{dE}{dt}$

$$\int G^{-1}(\dot{h})^2 dS \sim$$

$$\sim \frac{G^2}{R^2}(\dddot{D})^2 R^2 \cdot G^{-1} =$$

$$= G(\dddot{D})^2$$

More exact computation, taking into account angular integration:

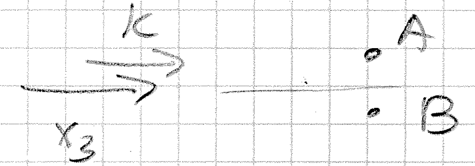
$$\frac{d\mathcal{E}}{dt} = -\frac{G_N}{45}\,\dddot{D}_{ij}\,\dddot{D}^{ij}$$

Binary systems: observed phenomenon!

pulsars (neutron stars)

Nobel prize, Taylor Joseph, H (1993)
Hulse Russel A.

Observation of gravitational waves:



A and B: in the plane orthogonal to x_3

distance between A & B:

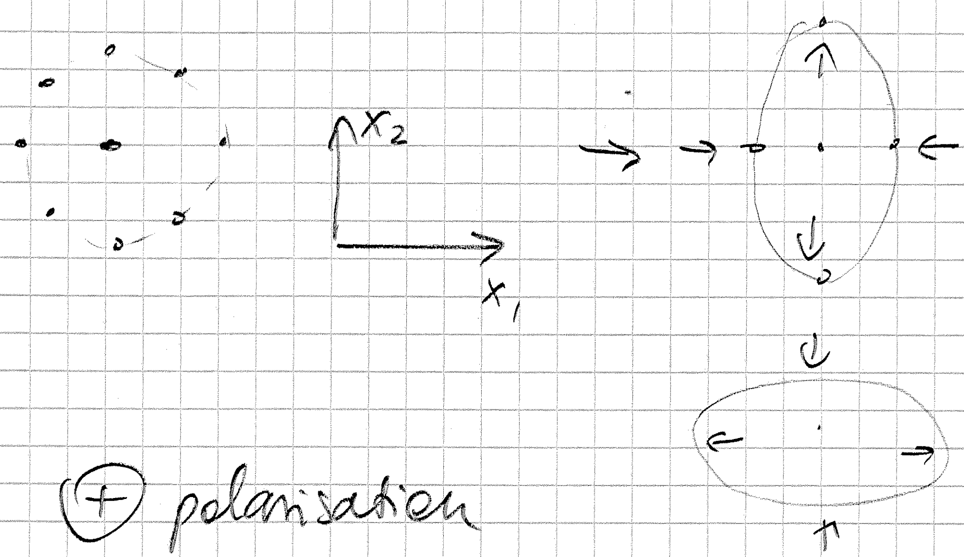
$$dl^2 = g_{ij}\,\Delta x^i \Delta x^j =$$

$$= h\left(\Delta x_1^2 - (\Delta x_2)^2\right) + h_{12}\,\Delta x_1 \Delta x_2 \cdot 2$$

$$+ \Delta x_1^2 + \Delta x_2^2$$

distance changes, and this potentially can be observed two types of polarisations:

(i) $h \neq 0$; $h_{12} = 0$:



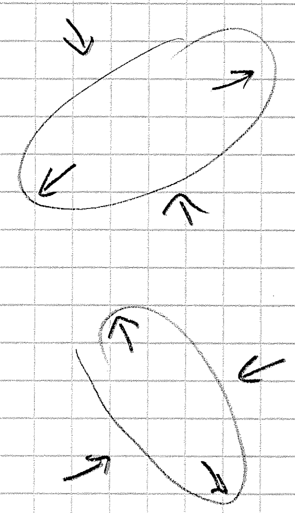
$t = t_1$

phase difference: π

$t = t_2$

(+) polarisation

(ii) $h = 0$; $h_{12} \neq 0$



(×) polarisation

45° difference between