Gravitational waves.

We should expect to have gravitational waves—perturbations of the metric—to exist in GR, in complete analogy with electrodynamics.

Weak gravitational waves:

\[ \delta g_{\mu \nu} = \psi + 8\pi \rho \gamma_{\mu \nu}, \quad 8\pi \rho \ll 1. \]

Wave equation in empty space:

\[ R_{\mu \nu} = 0, \quad \text{where } R_{\mu \nu} \text{ is expressed via } \delta g_{\mu \nu}. \]

Linear analysis: find \( R_{\mu \nu} \) as a function of \( \delta g_{\mu \nu} \), and keep only linear terms. We will get linear differential equations ⇒ we have superposition principle and may work in Fourier space.

Choose:

\[ \delta g_{\mu \nu} = \rho \gamma \exp(\imath k_{\mu \nu} x), \]

wave propagating in direction \( \mathbf{k} \).
Let us first consider this problem for electrodynamics:

\[ \begin{align*}
\n_n e^{i k_0 x^0} \\
\n\partial_n F^\mu{}_n = 0 \quad \text{for } \partial_\mu A_\mu = 0
\end{align*} \]

we get \((k^2 g_{\mu \nu} - k_\mu k_\nu) A^\nu = 0\)

Let us choose wave propagation in direction \( \hat{z} \): \( k_3 = k_0 = 0, \ k_1 = k_2 = 0 \)

Then for \( A^1 \) \( A^2 \) equations are:

\((\omega^2 - k^2) A^1 = 0; \ (\omega^2 - k^2) A^2 = 0\) \hspace{1cm} (1)

and the pair of equations for \( A^0, A^3 \)

gives \( \omega A_3 - k A_0 = 0 \) \hspace{1cm} (2)

Interesting (and very important) fact:

number of variables is 4 \((A_\mu)\) but

number of equations is 3 — one of

variables is arbitrary. Reason: gauge

invariance: \( A_\mu \) and \( A_\mu - \text{i} \kappa \partial_\mu \) are

equivalent. In Fourier space we have:

\( \omega \rightarrow \omega - \kappa \), \( A_1 \) and \( A_2 \) do not change,

\( A_0 \rightarrow A_0 - \omega \kappa \); \( A_3 \rightarrow A_3 - \kappa \omega \)
let us use this freedom and chose
\[ A_0 = 0 \] (can always be made if
initially \( A_0 \neq 0 \); make gauge transformation
with \( \lambda = A_0 / (i \omega) \)). Then, from eq. (2),
\( A_3 = 0 \).

Solutions for \( A_1, A_2 \) (which are gauge-invariant) are plane waves with \( \omega = \pm k \):
\[
\begin{cases}
A_2 = c_2 \exp \left[ -i k (z \pm t) \right] \\
A_1 = c_1 \exp \left[ -i k (z \pm t) \right] \\
A_0 = A_3 = 0
\end{cases}
\]
\( c_1 \) and \( c_2 \) correspond to 2
possible polarizations.

(\textit{massless spin 2 particle - photon, in quantum field theory})
For gravitational field:

\[ R_{\mu\nu} = \frac{1}{2} \left( k^2 h_{\mu\nu} - k_\alpha k_\mu h^\alpha_{\nu} - k_\alpha k_\nu h^\alpha_{\mu} + k_{\mu\nu} h_\alpha h^\alpha \right) \]

\[ R = \frac{1}{2} \left( k^2 h - k_\alpha k_\mu h^\alpha_{\mu} + k^2 h - k_\alpha k_\mu h^\alpha_{\mu} \right) \]

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \text{insert } R_{\mu\nu} \text{ and } R \text{ from above, replace } g_{\mu\nu} \text{ by } g_{\mu\nu} \]

All 10 equations are given on the next page.
We will get then 10 equations for
10 amplitudes $y_{n}$, listed below

For $G_{n} = R_{n} - \frac{1}{2} g_{n} k^{2}$ - E. tensor

00 : $(h_{11} + h_{22}) k^{2} = 0$ \( \checkmark \)
01 : $(h_{11} + h_{22}) \omega k = 0$ \( \checkmark \)
02 : $(-h_{20} k + h_{23} \omega) k = 0$ \( \checkmark \)
03 : $(-h_{10} k + h_{13} \omega) k = 0$ \( \checkmark \)
33 : $(h_{11} + h_{22}) \omega^{2} = 0$ \( \checkmark \)
32 : $(-h_{20} k + h_{23} \omega) \omega = 0$ \( \checkmark \)
31 : $(-h_{10} k + h_{13} \omega) \omega = 0$ \( \checkmark \)
22 : $h_{11} k^{2} + \omega (-2 h_{30} k + h_{33} \omega) + h_{11} (\omega^{2} - k^{2}) = 0$
21 : $h_{12} (\omega^{2} - k^{2}) = 0$
11 : $h_{11} k^{2} + \omega (-2 h_{30} k + h_{33} \omega) + h_{22} (\omega^{2} - k^{2}) = 0$

00; 03; 33 - Same
02; 32 - the same
01; 31 - the same

Visibly, 6 different equations for
10 unknowns

Independent 6 equations are given on
the next page.
3 equations \((00, 03, 33)\): 
\[ h_{11} + h_{22} = 0 \]

2 equations \((02, 32)\):
\[ -h_{20} k + h_{23} \omega = 0 \]

2 eq
\((01, 31)\):
\[ -h_{10} k + h_{12} \omega = 0 \]

1 eq
\((31)\):
\[ h_{12} (0^2 - k^2) = 0 \]

1 eq \([02-11]\):
\[ (h_{11} - h_{22})/0^2 - k^2 = 0 \]

1 eq \([00+11]\):
\[ h_{00} k^2 + \omega (2h_{30} k + h_{33} \omega) = 0 \]

4 variables cannot be determined.

Reason: the same as in Electrodynamics.

We have a freedom to make arbitrary coordinate transformations:

Coordinate transformations:
\[ x^\mu \rightarrow x'^\mu = x^\mu + \xi^\mu \] where \( \xi^\mu \) is small \( \Rightarrow \)

\[ g^\mu_\nu = \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x'^\nu}{\partial x^\beta} g^\alpha_\beta \Rightarrow \text{in linear order} \]
\[ h_{\mu\nu} \rightarrow h'_{\mu\nu} = \xi^\mu - \partial_\mu \xi^\nu - \partial_\nu \xi^\mu \]
Coordinate transformations:

\[ h_{00} \rightarrow h_{00} - 2i\omega \xi_0 \]

\[ h_{01} \rightarrow h_{01} - i\omega \xi_1 \]

\[ h_{02} \rightarrow h_{02} - i\omega \xi_2 \]

\[ h_{03} \rightarrow h_{03} - i\omega \xi_3 - i\kappa \xi_0 \]

\[ h_{13} \rightarrow h_{13} - i\kappa \xi_1 \]

\[ h_{23} \rightarrow h_{23} - i\kappa \xi_2 \]

\[ h_{33} \rightarrow h_{33} - i\kappa \xi_3 \]

without change:

\[ h_{11} \rightarrow h_{11} \]

\[ h_{22} \rightarrow h_{22} \]

\[ h_{12} \rightarrow h_{12} \]

So, \( h \) and \( h_{12} \) are physical - do not change if we change coord. system.

Let us use this freedom to put some components of \( h_{\nu} \) to zero:

\[ \xi_0 : \text{put } h_{00} \text{ to zero, } \xi_0 = (h_{00})_{\text{initial}} / (i\omega) \]

\[ \xi_3 : \text{put } h_{33} \text{ to zero, } \xi_3 = (h_{33})_{\text{initial}} / (i\kappa) \]

\[ \xi_2 : \text{put } h_{02} \text{ to zero, } \xi_2 = (h_{02})_{\text{initial}} / (i\omega) \]

\[ \xi_1 : \text{put } h_{10} \text{ to zero, } \xi_1 = (h_{10})_{\text{initial}} / (i\omega) \]
After this is made, our 6 equations are:

\[
\begin{align*}
    h_{11} + h_{22} &= 0 \\
    h_{23} &= 0 \\
    h_{13} &= 0 \\
    h_{12}(\omega^2 - k^2) &= 0 \\
    (h_{11} - h_{22})(\omega^2 - k^2) &= 0 \\
    h_{30} &= 0
\end{align*}
\]

\[\Rightarrow \quad h_{22} = -h_{11} = -h\]

\[
\begin{align*}
    h_{00} &= h_{33} = h_{01} = h_{02} = h_{03} = h_{13} = h_{23} = 0
\end{align*}
\]

\[\Rightarrow \quad h_{0}(\omega^2 - k^2) = 0\]

**Conclusion for gravitational wave 1/2:**

We have the following physical components:

\[
h_{\mu\nu} = \begin{pmatrix}
    0 & 0 & 0 & 0 \\
    0 & h & h_{2} & 0 \\
    0 & h_{2} & -h & 0 \\
    0 & 0 & 0 & 0
\end{pmatrix} \ e^{i\omega t - ikz}
\]

**Two polarizations:***

In quantum theory: spin 2, 2

polarizations:

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"Graviton" - particle, corresponding to gravitational wave
Emission of gravitational waves

we are here

 Relevant equations:

\[ G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} G_{\mu\nu} R = 8\pi G T_{\mu\nu} \]

For physical amplitudes, in linear approximation:

\[ \square h_{22} = 8\pi G T_{22} \]

\[ \square h = 8\pi G \frac{1}{2} [T_{22} - T_{44}] \]

we have already solved these type of equations in lectures on Electrodynamics.

Solution: via retarded Green's functions

\[ G(x, t, x', t') : \]

\[ h_{12}(x, t) = \int 8\pi G T_{12}(x', t') G(x, t; x', t') \, d^4x' \, dt' \]

\[ = \frac{2G}{R} \int T_{12}(x', t - |x-x'|) \frac{1}{(p - x')^2} \, d^3x' \approx \]

\[ \text{retarded time!} \]

\[ \approx \frac{2G}{R} \int d^3x' T_{12}(x', t - R) \]

valid if \( \frac{R}{c} \ll c \)

moving particles
Computation of integrals

Solve for: similar what we did with currents in electrodynamics.

\[
\frac{\partial \vec{E}}{\partial t} \cdot \vec{x_0} + \frac{\partial \vec{B}}{\partial t} \cdot \vec{x_0} = 0 \\
\frac{\partial \vec{E}}{\partial \vec{x_0}} - \frac{\partial \vec{B}}{\partial \vec{x_0}} = 0
\]

First eq: multiply by \(\vec{x_0}\) and integrate over space:

\[
\frac{2}{\partial \vec{x_0}} \int \vec{E} \cdot \vec{x_0} dV = \int \frac{\partial \vec{E}}{\partial \vec{x_0}} \cdot \vec{x_0} dV = \int \vec{F} \cdot \vec{d} \vec{x} = \\
\int \left[ \frac{\partial \vec{E}}{\partial \vec{x_0}} \right] dV
\]

vanished, Gauss theorem \(\Rightarrow\)

\[
\int \vec{E} \cdot \vec{d} \vec{V} = -\frac{1}{2} \frac{\partial}{\partial \vec{x_0}} \int \left( \vec{E} \cdot \vec{x_0} + \vec{B}_0 \cdot \vec{x_0} \right) dV
\]

Second eq: multiply by \(\vec{x_0} \cdot \vec{x}\), and integrate \(\Rightarrow\)

\[
\frac{2}{\partial \vec{x_0}} \int \vec{E} \cdot \vec{x_0} \cdot \vec{x} dV = -\int \vec{F} \cdot \vec{x} dV = \int \frac{\partial \vec{B}}{\partial \vec{x_0}} \cdot \vec{x} dV
\]

\[
\int \vec{B} \cdot \vec{d} \vec{V} = \frac{1}{2} \left( \frac{\partial}{\partial \vec{x_0}} \right)^2 \int \vec{E} \cdot \vec{x_0} \cdot \vec{x} dV
\]

\(p(\vec{x}) - \text{mass distribution}\)
Quadropole moment of mass distribution:

\[
\mathbf{D} = \int \rho(x) \left[ 3x^2 \mathbf{a} - r^2 \mathbf{a} \right] dV \\
\Rightarrow
\]

\[
h_{12} \approx \frac{2G}{3R} \mathbf{\ddot{D}}_{12} ; \quad h = \frac{2G}{3R} \cdot \frac{1}{2} (\mathbf{\dddot{D}}_{12} - \mathbf{\ddot{D}}_{11})
\]

Energy loss, order of magnitude estimate:

Energy density of gravitational wave

since \( h \) is analogue of vector potential, energy density of gravitational wave must contain \( (h) \) \( \text{or} \) \( (h_{12}) \) \( \text{or} \) \( (\mathbf{w})^2 \)

energy density: \( G \mathbf{w}^2 \Rightarrow \)

\[
\mathcal{E} \propto G^{-1} (h)^2
\]

heat of energy \( \sim \frac{d\mathcal{E}}{dt} \)

\[
\int G^{-1} (h)^2 ds \sim
\]

\[
\sim \frac{G^2}{R^2} \left( \mathbf{\dddot{D}} \right)^2 R^2 \cdot G^{-1} = \]

\[
= G (\mathbf{\dddot{D}})^2
\]
More exact computation, taking into account angular integration:

\[
\frac{dE}{dt} = \frac{G M}{45} \bar{D}_{ij} \bar{D}_{ij}
\]

Binary systems: observed phenomenon.

Pulsars (neutron stars)


Hulse Russell A.

Observation of gravitational waves:

\[ k \]

\[ A \]

\[ B \]

A and B: in the plane orthogonal to \( x_3 \)

distance between \( A \) \& \( B \):

\[
dl^2 = g_{ij} dx^i dx^j = \\
= \hbar (\Delta x_2^2 - (\Delta x_2)^2) + h_{12} \Delta x_1 \Delta x_2 \cdot 2 \\
+ \Delta x_2^2 + \Delta x_2^2
\]

distance changes, and this potentially can be observed.
two types of polarisations:

(i) $h \neq 0; h_{zz} = 0$

\[ t = t_1 \]

\[ \oplus \text{polarisation} \]

(ii) $h = 0; h_{zz} \neq 0$

\[ t = t_2 \]

\[ \bigotimes \text{polarisation} \]

$45^\circ$ difference between