Lecture # 13

Main points of # 12:

Found equations of motion:

proper time τ	time t
$\theta = \frac{\pi}{2}$	$\theta = \frac{\pi}{2}$
$m r^2 \frac{d\varphi}{d\tau} = J$	$r^2 \frac{d\varphi}{dt} = \frac{JB}{E}$
$A m^2 \left(\frac{dr}{d\tau}\right)^2 - \frac{E^2}{B} + \frac{J^2}{r^2} = -m^2$	$\frac{A}{B^2} E^2 \left(\frac{dr}{dt}\right)^2 - \frac{E^2}{B} + \frac{J^2}{r^2} = -m^2$

$$\boxed{d\tau = \frac{m}{E} B \, dt}$$

Light deflection: $\Delta\varphi \approx \frac{4MG}{r_0} = \frac{2r_g}{r_0}$

Perihelium of mercury: $\Delta\varphi \approx 6\pi \frac{MG}{L} = 3\pi \frac{r_g}{L}$,

$$L = \frac{1}{2}\left(\frac{1}{r_+} + \frac{1}{r_-}\right)$$

Today: black holes.

Let us start with motion of particles in S. metric and <u>do not</u> assume that $\frac{r_g}{r} \ll 1$

for $m \neq 0$ let us use proper time, and write equation for r as:

$$\left(\frac{dr}{d\tau}\right)^2 + \frac{1}{A} + \frac{J^2}{Am^2r^2} = \frac{E^2}{m^2}$$

or $$\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + \frac{1}{2}\left(1 - \frac{r_g}{r}\right)\left(1 + \frac{J^2}{m^2r^2}\right) = \frac{1}{2}\frac{E^2}{m^2}$$

mechanical analog: moving of particle in 1-dimension, $r \in (0, \infty)$ in "potential"

$$U = \frac{1}{2}\left(1 - \frac{r_g}{r}\right)\left(1 + \frac{J^2}{m^2r^2}\right)$$

with total "energy" $= \frac{1}{2}\frac{E^2}{m^2}$

For massless particle, let us use new time variable λ: $d\lambda = B\,dt$, to simplify equations:

$$r^2 \frac{d\varphi}{d\lambda} = \frac{J}{E}$$

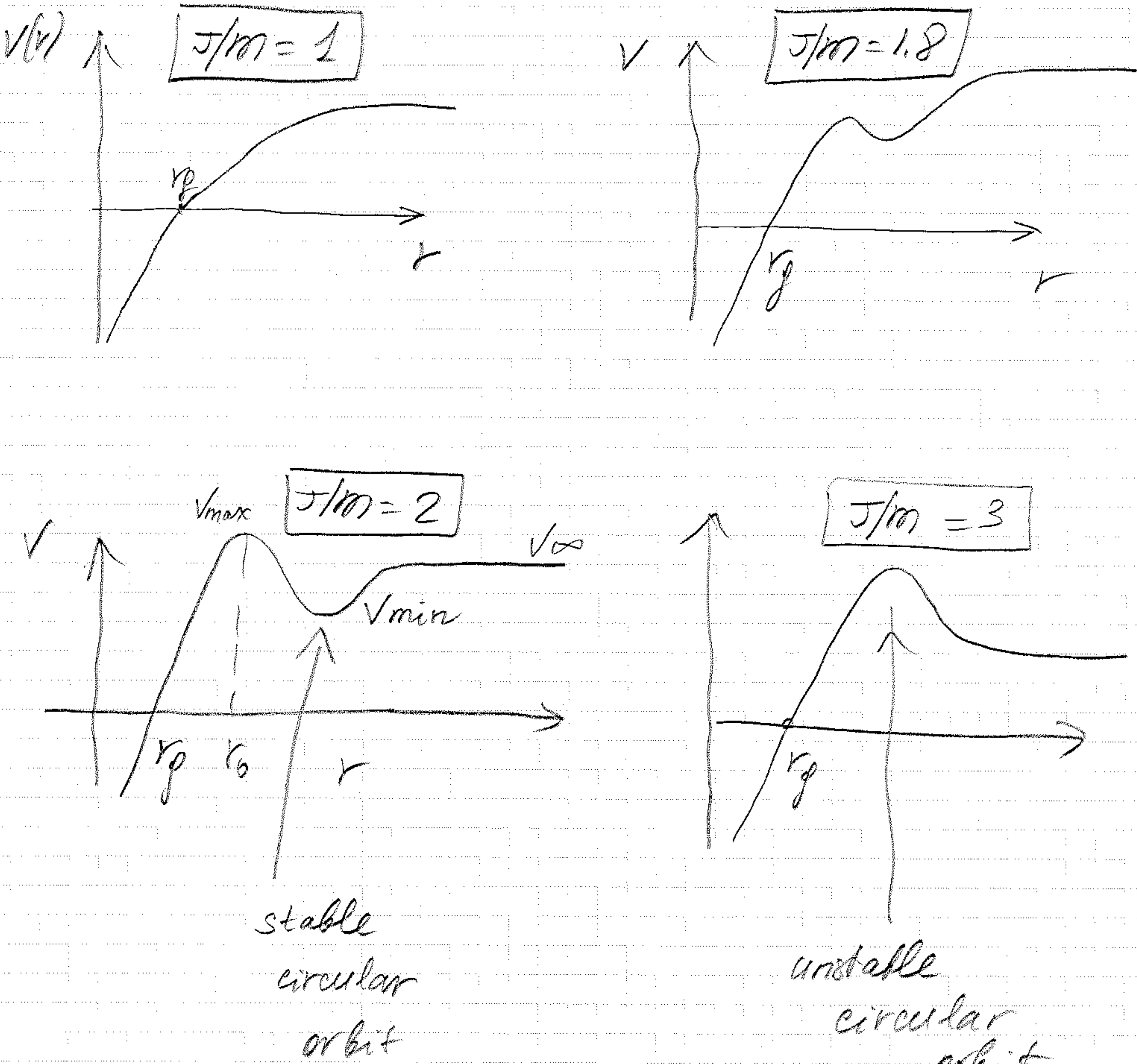
$$E^2 A \left(\frac{dr}{d\lambda}\right)^2 - \frac{E^2}{B} + \frac{J^2}{r^2} = 0,$$

or
$$\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + \frac{1}{2}\underbrace{\left(\frac{J^2}{E^2 A r^2} - 1\right)}_{\frac{J^2}{E^2}\left(1-\frac{r_0}{r}\right)-1} = 0$$

or
$$\underbrace{\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2}_{\text{kinetic energy}} + \underbrace{\frac{1}{2}\left(1-\frac{r_0}{r}\right)\cdot\frac{J^2/E^2}{r^2}}_{\text{"potential energy"}} = \underbrace{\frac{1}{2}}_{\text{total energy}}$$

Radial motion, $m \neq 0$:

Potential as a function of r, and J/m:



massive case:

3 possible orbits: (for topology corresponding to $J/m = 2$, and $r \to \infty$ at $\tau \to -\infty$

(i) $V_{min} < \frac{1}{2}\frac{E^2}{m^2} < V_\infty$;

closed orbits

(ii) $V_\infty < \frac{E^2}{2m^2} < V_{max}$ —

"scattering", open orbits

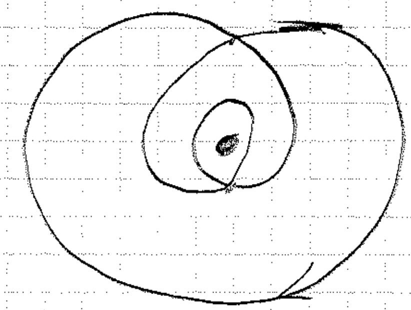
(iii) $\frac{E^2}{2m^2} > V_{max}$ — fall to the center (absent in Newtonian theory, where $r_g = 0$

If $r(0) < r_0$, and $\frac{E^2}{2m^2} > V_{max}$,

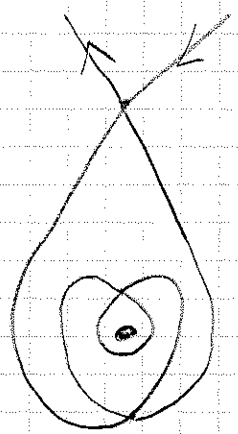
trajectory will go to ∞

if $\frac{E^2}{2m^2} < V_{max}$ — fall to the center

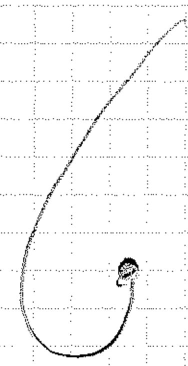
Possible trajectories, for $L/r_g = 2$



, between V_∞ and V_{min}



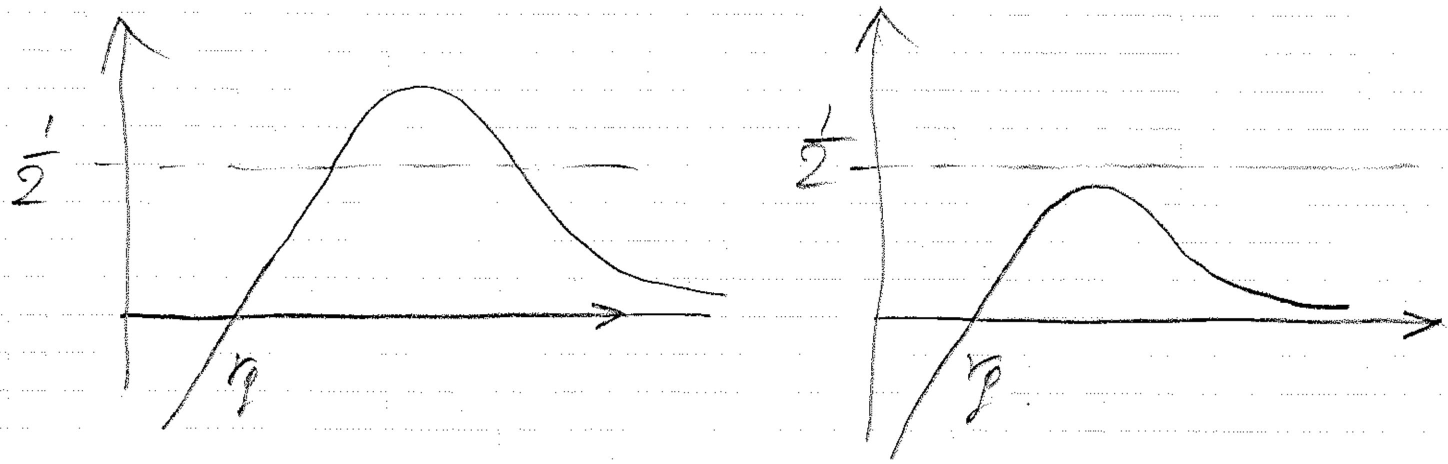
between V_{max} and V_∞

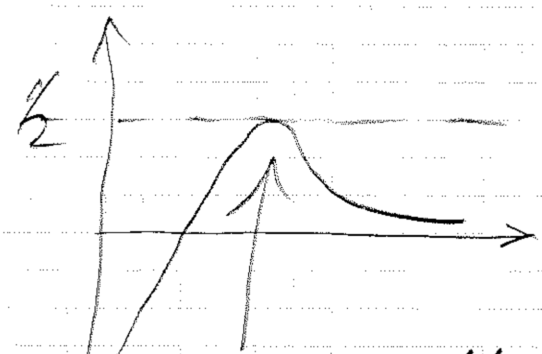


above V_{max}

below V_{max}, but to the left of maximum

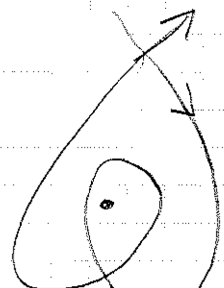
Radial motion, $m=0$



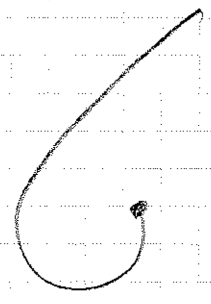


unstable circular orbit

Possible trajectories



– light deflection



falling to the center

massless case

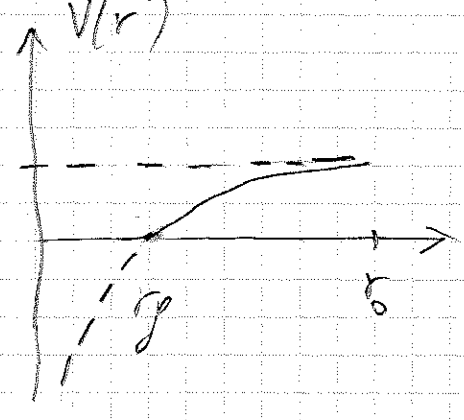
(i) $V_{max} > \frac{1}{2}$: scattering, light deflection

(ii) $V_{max} < \frac{1}{2}$: fall to the center
absent in Newtonian
theory, where $r_g = 0$

For all cases: if particle reach from infinity r_g, it will never return and fall into singularity

Consider now radial motion, with $J=0$, massive particle

$$\left(\frac{dr}{d\tau}\right)^2 = \frac{E^2}{m^2} - \left(1 - \frac{r_g}{r}\right),$$



$\frac{dr}{d\tau} = 0$ at $r = r_0$

$$\Rightarrow \left(\frac{dr}{d\tau}\right)^2 = \left(1 - \frac{r_g}{r_0}\right) - \left(1 - \frac{r_g}{r}\right)$$

$$= r_g\left(\frac{1}{r} - \frac{1}{r_0}\right);$$

$$\frac{dr}{d\tau} = \pm\sqrt{\frac{(r_0 - r)}{r r_0} r_g}$$ choose $-$, since r is decreased $\Rightarrow$

$$\frac{dr}{\sqrt{r_g}}\sqrt{\frac{r r_0}{r_0 - r}} = -d\tau;$$

$$\frac{dr}{r_0}\sqrt{\frac{r_0}{r_g}}\sqrt{\frac{\frac{r}{r_0}}{1 - \frac{r}{r_0}}} = -\frac{d\tau}{r_0}, \quad \frac{r}{r_0} = z$$

$$dz\sqrt{\frac{z}{1-z}} = -\frac{d\tau}{r_0}\sqrt{\frac{r_g}{r_0}}$$

$\Rightarrow$ finite time to reach r_g: τ_g and finite time to reach singularity at $r=0$: $\tau_0 = \frac{\pi}{2} r_0 \sqrt{\frac{r_0}{r_g}}$, $\tau_g < \tau_0$

time, necessary to reach r_g for observer at r_0, $r_0 \gg r_g$:

$$\frac{E}{m}\int dt = \int_0^\tau \frac{d\tau}{B(r)} = \int \frac{1}{B(r)}\frac{d\tau}{dr}dr =$$

$$= \int_0^r \frac{1}{\left(1-\frac{r_g}{r}\right)} \cdot \sqrt{\frac{r r_0}{(r_0 - r) r_g}}\, dr$$

for $r \to r_g$: singularity $\Rightarrow$

$$\frac{E}{m} t \approx \int_0^r \frac{r_g}{r - r_g} \sqrt{\frac{\cancel{r_g} r_0}{(r_0 - r_g)\cancel{r_g}}}\, dr \approx$$

$$\approx r_g \left[\log \frac{r_g}{r - r_g} + Const\right]$$

So, when $r \to r_g$ $t \to \infty$ — for external observer the test particle will <u>never</u> reach r_g !!!

proper time to reach singularity after crossing $r = r_g$:

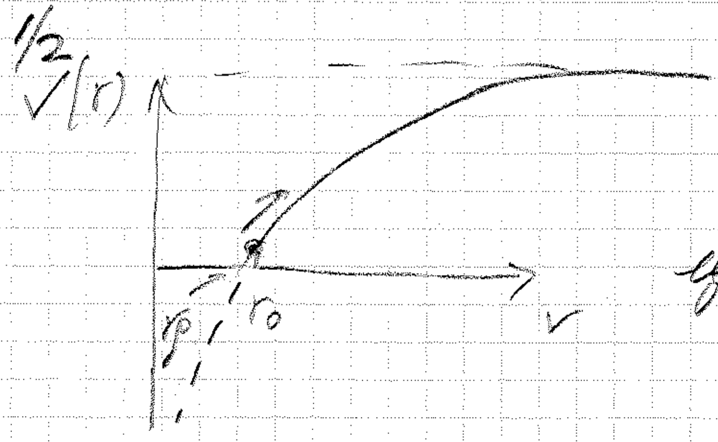
$$t = \frac{\pi}{2} r_g = \pi G_N M$$

Escape velocity:

Let us study the case when a star has radius $r > r_g$, and then take a limit $r \to r_g$

Example (close) - neutron stars, $m \sim M_\odot$, $r \sim 10\,km$

Again, our potential:



escape velocity: equation

$$\frac{dr}{d\tau} = + \sqrt{\frac{E}{m} - \left(1 - \frac{r_g}{r}\right)}$$

$$r \to \infty \Rightarrow \frac{dr}{d\tau} = \sqrt{\frac{E}{m} - 1} \Rightarrow$$

for escape, choose $E = m \Rightarrow$

$$\frac{dr}{d\tau} = \sqrt{\frac{r_g}{r}} \quad \text{if } r = r_0 \Rightarrow$$

$$\frac{dr}{d\tau} = \sqrt{\frac{r_g}{r_0}}, \text{ and } \frac{dr}{d\tau} \to 1 \text{ for } r = r_g$$

Therefore escape velocity $\to$ speed of light when $r \to r_g$.

$\frac{dr}{d\tau}$ is indeed the velocity measured by observer on a star:

true time: $\Delta T = \sqrt{B}\, dt$

true distance: $\Delta X = \frac{1}{\sqrt{B}}\, dr$

From $ds^2 = B\,dt^2 - \frac{1}{B}\,dr^2$

speed: $\frac{\Delta X}{\Delta T} = \frac{1}{B}\frac{dr}{dt} = \frac{1}{B}\frac{dr}{d\tau}\frac{d\tau}{dt} =$

$$= \frac{1}{B}\cdot\frac{dr}{d\tau}\cdot\frac{m}{E}B = \frac{dr}{d\tau} \to 1$$

<u>Gravitational red-shift:</u>

From lecture # 6.

put an emmitor with internal frequency ν_0 at $r \to r_g$

what is the frequency observed by external observer [very far from the star]

From lecture # 6:

$d\tau = dt \sim \frac{1}{\nu_{obs}}$ $\qquad$ $d\tau = \sqrt{B(r)}\, dt \sim \frac{1}{\nu_0}$

$$\Downarrow$$

$$\frac{1}{\nu_{obs}} = \frac{1}{\sqrt{B(r)}\,\nu_0} \Rightarrow \nu_{obs} = \nu_0\sqrt{1-\frac{r}{r_g}}$$

So, $\nu_{obs} \to 0$ for $r \to r_g$

Conclusion: nothing comes out from a star if $r \to r_g$

$\Downarrow$

<u>Black</u> <u>hole</u>

nothing out — everything in

Can be shown: Singularity at $r = r_g$ is associated with the choice of coordinates - and can be removed.

surface: $r = r_g$ - event horison (nothing is seen behind!)

Mechanism for BH production: star collapse. BH & neutron stars were predicted in 30th mainly by Oppenheimer and collaborators.

Method: consider E. equations inside star - give E-M tensor (energy momentum), and find solutions. Check whether the solution is stable. Many effects are essential: nuclear & and fusion reactions, neutrinos, etc.

Collapse: one has to consider time-dependent solutions ....

For realistic cases very complicated

Doable case - spherically symmetric matter with $p=0$ (pressure)

(no supernova explosion in spherically-symmetric numerical simulations)

Evidence for Black holes –

it is believed that there is BH in any center of galaxy

Our galaxy – Milky way,

Sagittarius A^* – our BH ($Sgr\ A^*$)

$r = 22 \cdot 10^6\ km$ [half of diameter of Mercury orbit]

$M = 3.7 \cdot 10^6\ M_\odot$

Quantum Effect: Hawking radiation

In fact, BH emit light as if they were Black bodies at temperature

$$T_H = \frac{1}{4\pi r_g} \qquad r_g = 2MG_N$$

to small to be observed for any astrophysical BH