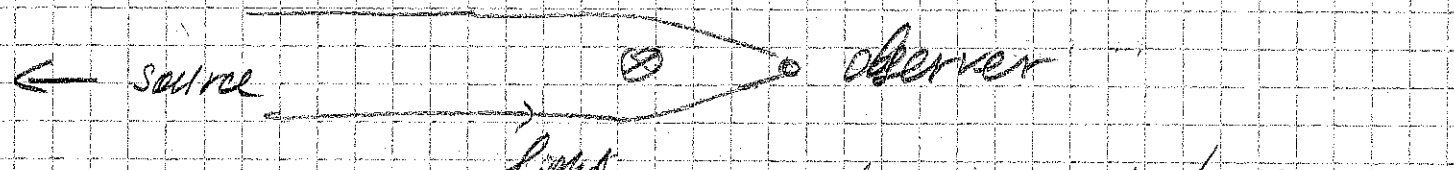
Deflection can produce multiple images :

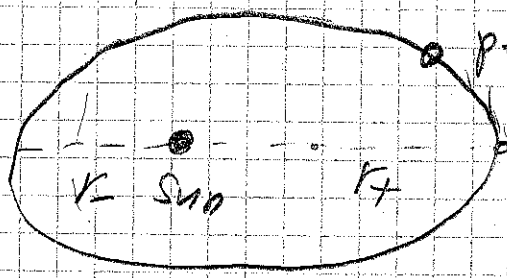


two images here.

Method for study of matter distribution in the Universe (G. Meylan & Courbin)

Closed orbits: Mercury perihelium

Newton gravity:



in r_- and r_+ we have $\frac{dr_-}{d\varphi} = \frac{dr_+}{d\varphi} = 0$ (*)

in GR the orbit is more complicated but (*) holds, of course

Integrals of motion:

$$\frac{1}{r_\pm^2} - \frac{m^2}{J^2} + \frac{E^2}{B(r_\pm)J^2} = 0$$

We can find 2 integrals of motion:

$$\frac{m^2}{E^2} = \frac{\frac{r_+^2}{B(r_+)} - \frac{r_-^2}{B(r_-)}}{r_+^2 - r_-^2}$$

$$\frac{J^2}{E^2} = \frac{\frac{1}{B(r_+)} - \frac{1}{B(r_-)}}{\frac{1}{r_+^2} - \frac{1}{r_-^2}}$$

change of angle, if we go from r_- to r_+:

$$2(\varphi(r_+) - \varphi(r_-))$$

if this is a closed curve, then it should be equal to (2π).

In general, the shift of perihelium is given by

$$\Delta\varphi = 2|\varphi(r_+) - \varphi(r_-)| - 2\pi,$$

where

$$\varphi(r_+) - \varphi(r_-) = \int_{r_-}^{r_+} \frac{\sqrt{A}\,dr}{r^2} \left[\frac{r_-^2(B^{-1}(r) - B^{-1}(r_-)) - r_+^2(B^{-1}(r) - B^{-1}(r_-}{r_+^2 r_-^2 (B^{-1}(r_+) - B^{-1}(r_-))} - \frac{1}{r^2} \right]^{-1/2}$$

Computation: again, expand with respect to small parameter(s) $\frac{GM_\odot}{r}$:

$$A = 1 + 2\frac{MG}{r} + \dots$$

$$B = 1 - \frac{2MG}{r}$$

Computation is left for exercise:

$$\Delta\varphi = 6\pi \frac{MG}{L},$$

gives nonzero result even for circular orbit, but then it is unobservable

where $\frac{1}{L} = \frac{1}{2}\left(\frac{1}{r_+} + \frac{1}{r_-}\right)$

$\Delta\varphi > 0$: perihelium moves in the same direction as the particle

Mercury: (why Mercury – L is smaller)

$L = 55.3 \cdot 10^6$ km

$MG = 1.475$ km

$\Delta\varphi = 0.1038''$ for one turn

100 years: 415 turns $\Rightarrow$

$\Delta\varphi \Rightarrow N\Delta\varphi = 43.03''$ for hundred years

observations exist starting from 1765

Comparison with observations is non-trivial, since there are other reasons for change of the orbit:

Newton theory gives

$\Delta\varphi = 532''$ (for 100 years) due to perturbations related to Venus, Earth, Jupiter, so that GR contribution is just ~ 10%

Other possible effects: Sun is not spherically symmetric.

Yet another test (we will not analyse it) – delay of radar echo.

travel time Earth → planet → earth