Lecture # 12

Main points of lecture —

- shown that non-relativistic limit of E. equations gives Newton gravity theory
- found general spherically symmetric static metric

$$ds^2 = B(r)dt^2 - A(r)dr^2 - r^2 d\Omega^2$$

A and B – arbitrary

- Solved E. equations $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ for spherically symmetric metric and found:

$$A \cdot B = 1;$$

$$B = 1 - \frac{2MG}{r}$$

M – mass of the body

Schwarzschild solution.

Today: motion in spherically symmetric metric

Remarks: validity of non-relativistic approximation: $A \approx B \approx 1 \Rightarrow$

$$\frac{2MG}{r} \ll 1.$$

- combination $r_g = 2MG$ is called gravitational radius, or Schwarzschild radius
- Something happens for $r < r_g$:
  for $r = r_g$ $A = \infty$, $B = 0$, and
  for $r < r_g$ A and B change signes
  This will be discussed later, in study of black holes.

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How to discuss motion in S. metric.

option 1 - use geodesic equation,

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda} \frac{dx^\nu}{d\tau} \frac{dx^\mu}{d\tau} = 0$$

option 2 - use action principle

Let us take 2 - to see the power of action method. Consider massive particle

$$S = -\frac{1}{2} m \int g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} d\tau,$$

τ — proper time

$$S = -\frac{1}{2} m \int d\tau \left[B\left(\frac{dt}{d\tau}\right)^2 - A\left(\frac{dr}{d\tau}\right)^2 - \sin^2\theta\, r^2 \left(\frac{d\varphi}{d\tau}\right)^2 - r^2 \left(\frac{d\theta}{d\tau}\right)^2 \right]$$

equations of motion – variation of the action with respect to t, r, φ, θ.

Note: no back reaction is taken into account: metric is assumed to be fixed!

$$\frac{\delta}{\delta t}: \quad m \frac{d}{d\tau}\left[B \frac{dt}{d\tau}\right] = 0$$

$$\frac{\delta}{\delta \varphi}: \quad m \frac{d}{d\tau}\left[r^2 \sin^2\theta \frac{d\varphi}{d\tau}\right] = 0$$

$$\frac{\delta}{\delta \theta}: \quad m \frac{d}{d\tau}\left[r^2 \frac{d\theta}{d\tau}\right] = 0$$

$$\frac{\delta}{\delta r}: \quad m \frac{d}{d\tau}\left[A \frac{dr}{d\tau}\right] + \frac{1}{2}\frac{\partial B}{\partial r}\left(\frac{dt}{d\tau}\right)^2 - \frac{1}{2}\frac{dA}{dr}\left(\frac{dr}{d\tau}\right)^2 - \frac{2r\sin^2\theta}{2}\left(\frac{d\varphi}{d\tau}\right)^2 - \frac{2r}{2}\left(\frac{d\theta}{d\tau}\right)^2 = 0$$

(in fact, we will not need equation for r). I will even not write it!

from eq for φ and θ we may conclude that motion occurs in a plane, e.g. $\theta = \frac{\pi}{2} = \text{Const} \Rightarrow$ eq for θ is satisfied and others are simplified:

$$m\,\frac{d}{d\tau}\left[B\,\frac{dt}{d\tau}\right] = 0 \qquad (*)$$

$$m\,\frac{d}{d\tau}\left[r^2\,\frac{d\varphi}{d\tau}\right] = 0 \qquad (**)$$

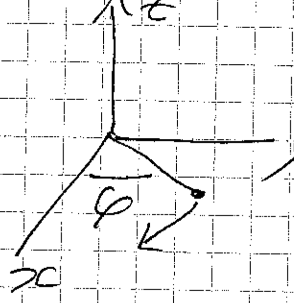
$$-\frac{1}{2}A'\left(\frac{dr}{d\tau}\right)^2 + \frac{d}{d\tau}\left[A\,\frac{dr}{d\tau}\right] + \frac{B'}{2}\left(\frac{dt}{d\tau}\right)^2 - \frac{2r}{2}\left(\frac{d\varphi}{d\tau}\right)^2 = 0 \qquad (***)$$

(we will not need equation for r).

From $(*)$ and $(**)$ we find two integrals of motion:

$$m\,B\,\frac{dt}{d\tau} = \text{Const},$$

$$m\,r^2\,\frac{d\varphi}{d\tau} = \text{Const} \qquad \text{angular momentum conservation!}$$



angular momentum: $\vec{p}\times\vec{x} = \hat{z}\cdot m\,r^2\frac{d\varphi}{dt}$ in non-relativistic mechanics

Denote $r^2 \frac{d\varphi}{d\tau} \equiv J/m$

Consider $B\frac{dt}{d\tau}$ at large distances, $r \to \infty$

Then $B \to 1$, $d\tau = (1-v^2)^{1/2} dt$, and

$B\frac{dt}{d\tau} \to 1 \cdot \frac{1}{(1-v^2)^{1/2}}$ - looks like

energy of the particle (up to mass m).

So, let us denote $B\frac{dt}{d\tau} = E/m$,

E has physical sence as energy of massive particle at $r \to \infty$

Now, use the fact that τ is proper time:

$$g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 1 :$$

$$B\left(\frac{dt}{d\tau}\right)^2 - A\left(\frac{dr}{d\tau}\right)^2 - r^2\left(\frac{d\varphi}{d\tau}\right)^2 = 1 \Rightarrow$$

$$A\left(\frac{dr}{d\tau}\right)^2 - \frac{E^2}{m^2 B} + \frac{J^2}{m^2 r^2} = -1 \qquad (\alpha)$$

It is easy to show that (***) is just a consequence of this equation - just find a derivative of it!

Summary: motion is given by equations

$$\begin{cases} r^2 \dfrac{d\varphi}{d\tau} = J/m, \quad \theta = \dfrac{\pi}{2}, \\ A\left(\dfrac{dr}{d\tau}\right)^2 - \dfrac{E^2}{Bm^2} + \dfrac{J^2}{r^2 m^2} = -1, \end{cases} \qquad (\tau)$$

if proper time is used, or

$$\begin{cases} r^2 \dfrac{d\varphi}{dt} = \dfrac{JB}{E}; \quad \theta = \dfrac{\pi}{2}, \\ \dfrac{E^2}{m^2} \dfrac{A}{B^2}\left(\dfrac{dr}{dt}\right)^2 - \dfrac{E^2}{m^2 B} + \dfrac{J^2}{m^2 r^2} = -1 \\ d\tau^2 = \dfrac{B^2 dt^2}{E^2} \cdot m^2 \end{cases} \qquad (t)$$

if time t is used.

Remark: for massless particles, like photon, we cannot use proper time, but can use someother parameter, like t. Equations (t) admit a smoth limit when $m \to 0$:

$$\begin{cases} \theta = \dfrac{\pi}{2}; \quad r^2 \dfrac{d\varphi}{dt} = \dfrac{JB}{E}; \\ E^2 \dfrac{A}{B^2}\left(\dfrac{dr}{dt}\right)^2 - \dfrac{E^2}{B^2} + \dfrac{J^2}{r^2} = 0 \\ d\tau^2 = 0 \end{cases}$$

Equation for orbits (trajectories)

r as a function of φ or vice-versa

$$dr = \frac{r^2 d\varphi}{J} m;$$

$$A\left(\frac{dr}{r^2 d\varphi}\frac{J}{m}\right)^2 - \frac{E^2}{Bm^2} + \frac{J^2}{r^2m^2} = -1;$$

$$\frac{A}{r^4}\,dr^2 + d\varphi^2\left[\frac{1}{r^2} + \frac{m^2}{J^2} - \frac{E^2}{BJ^2}\right] = 0$$

$$\text{and } \varphi = \pm \int \frac{\sqrt{A}\,dr}{r^2\left[\frac{E^2}{BJ^2} - \frac{m^2}{J^2} - \frac{1}{r^2}\right]^{1/2}} \qquad (\varphi)$$

This gives complete solution to the problem of motion in isotropic metric.

relations between notations: (old: last year)

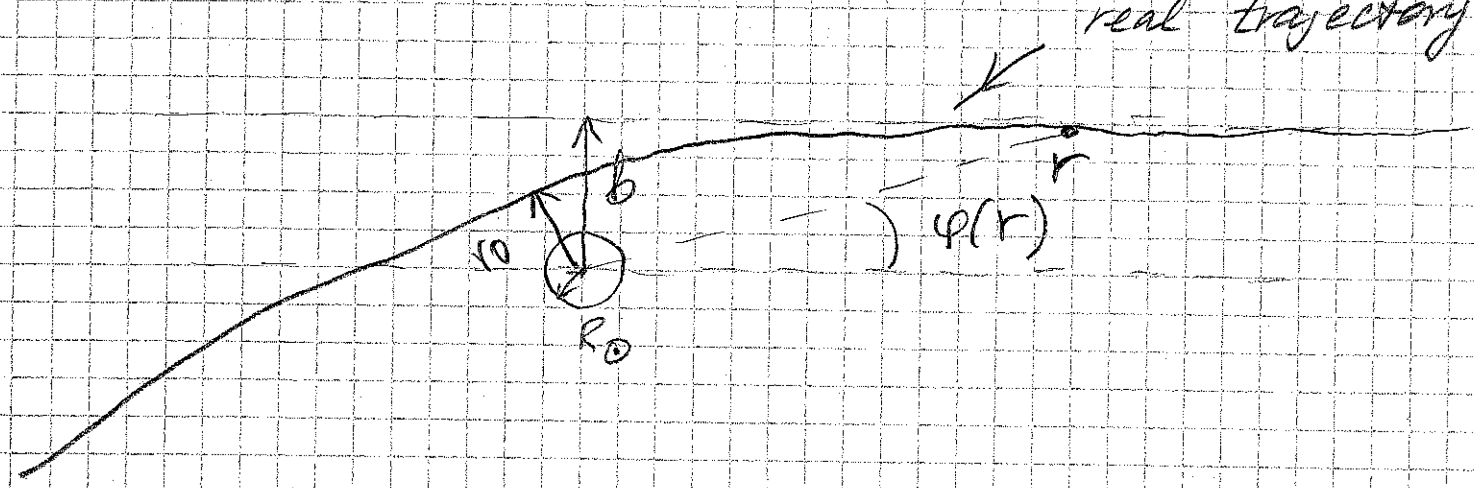
$$\frac{E^2_{new}}{m^2} = \frac{1}{E_{old}} \Rightarrow E_{new} = \frac{m}{\sqrt{E_{old}}}\,;\; E_{old} = \frac{m^2}{E^2_{new}}$$

$$\frac{J_{new}}{E_{new}} = J_{old} \Rightarrow J_{new} = \frac{J_{old}}{\sqrt{E_{old}}}\,;\; J_{old} = \frac{J_{new}}{E_{new}}$$

For light trajectories: just put $m=0$ in equation (φ)

Open orbits: light bending

Consider massive particle (we will take a limit $m \to 0$ later)



$R_\odot$: sun radius

b : impact parameter

$r \to \infty$: $A(r) \to 1$; $B(r) \to 1 \Rightarrow$

metric is flat, particle moves with constant velocity v

$$b \approx r \sin\varphi \approx r\varphi \quad \text{(we take } \varphi \text{ small)}$$

$$-v \approx \frac{d}{dt}(r\cos\varphi) \approx \frac{dr}{dt}$$

This solution satisfy our equations $(*)$ and give integrals of motion

$$J = bv/\sqrt{1-v^2}$$

$$E = m/\sqrt{1-v^2}$$

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Let r_0 is a minimal distance from trajectory to the sun $\Rightarrow$

$$\frac{dr}{d\varphi} = 0 \text{ @ } r_0 \text{, so, from (traject)}$$

we get

$$\frac{J^2}{r_0^2} - \frac{E^2}{B(r_0)} + m^2 = 0 \Rightarrow J = \frac{r_0 m}{\sqrt{1-v^2}} \left[\frac{1}{B(r_0)} - 1 + v^2\right]^{1/2}$$

(orbit) gives then:

$$\varphi(r) = \int_r^\infty \frac{\sqrt{A}\,dr}{r^2\left(\frac{1}{r_0^2}\left[\frac{1}{B(r)} - 1 + v^2\right]\left[\frac{1}{B(r_0)} - 1 + v^2\right]^{-1} - \frac{1}{r^2}\right)^{1/2}}$$

(if $r \to \infty$ $\varphi(r) \to 0$, as it should)

Change of angle (no bending: π)

$$\Delta\varphi = 2\varphi(r_0) - \pi$$

For photon $v = 1 \Rightarrow$

$$\varphi(r) = \int_r^\infty \frac{\sqrt{A}\,dr}{r\left[\left(\frac{r}{r_0}\right)^2\left(\frac{B(r_0)}{B(r)}\right) - 1\right]^{1/2}}$$

Computation:

$$A(r) \approx 1 + 2\frac{MG}{r}$$

$$B(r) \approx 1 - 2\frac{MG}{r}$$

then expand considering $\frac{MG}{r}$ as small parameter

Result (computation - in exers)

$$\Delta\varphi = \frac{4MG}{r_0}$$

in this order impact parameter coinsides with r_0, $b = r_0$

Numerical values:

$M_\odot = 1.87 \cdot 10^{33}\, g$, $MG = 1.475\, km$

for r_0 we can take sun radius,

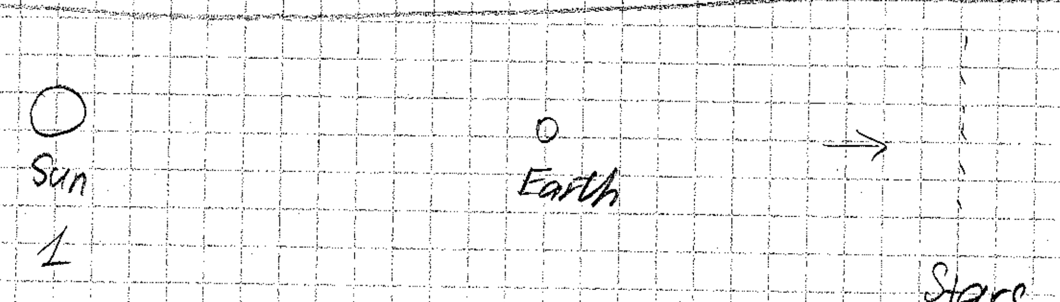
$R_\odot = 6.95 \cdot 10^5\, km$

$$\Delta\varphi = \frac{R_\odot}{r_0}\,\theta_\odot\,, \quad \theta_\odot = 1.75''$$

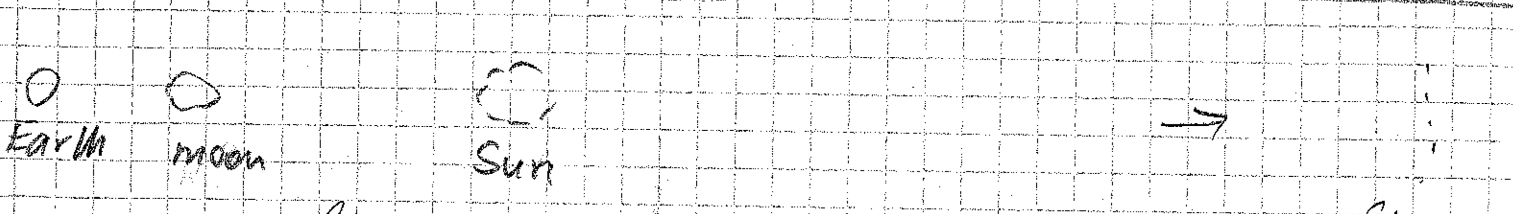
$\left[\begin{array}{l} {}^\circ - \text{degree} \\ {}' - \text{minute} \\ {}'' - \text{second} \end{array} \right]$

How to observe this?

Compare 2 observation of stars:



Observ # 1



observation # 2