Lecture # 7. ①

main points of # 6.

discussed kinetic equations and freezing approximation:

find $\Gamma(T^*)H(T^*) = 1 \Rightarrow$

$$N = \begin{cases} N_{eq}, & \text{if } \Gamma(T) > H(T) \\ N_{eq}(T^*)\left(\frac{R(T^*)}{R(T)}\right)^3, & \text{if } \Gamma(T) < H(T), \end{cases}$$

and particle is stable

Γ: only takes into account reactions that change N

Today:

- entropy conservation
- demonstration that we have indeed this solution
- decoupling of photons

Entropy conservation in expanding Universe.

For simplicity: closed univers.

Let S is the total entropy

Consider expansion of $\frac{1}{S}\frac{dS}{dt}$ with respect to

$H = \frac{1}{a}\frac{da}{dt}$:

$$\frac{1}{S}\frac{dS}{dt} = A + BH + CH^2 + \dots$$

$A = 0$, since if $H = 0$, then $\frac{dS}{dt} = 0$

(we are already in the state of thermal equilibrium)

$B = 0$, since $\frac{dS}{dt} > 0$, but we can change the sign of H

$\Rightarrow \frac{dS}{dt} = CH^2$. from dimensional reasons:

$C = s \Rightarrow \quad C \simeq \tau_0$ – reaction time $\Rightarrow$

$$\frac{1}{S}\frac{dS}{dt} = \tau_0 H^2 \Rightarrow \quad \frac{\delta S}{S} = \int_{t_1}^{t_2} \tau_0(t) H^2(t)\, dt$$

$$= O(\tau_0 H) \int H(t)\, dt = O(\tau_0 H) \log \frac{a(t_2)}{a(t_1)}$$

$\ll 1$, as $\tau_0 H \ll 1$

Solution to kinetic equation

Step # 1 : consider integrated version of (*) - (momentum dependence is not very interesting)

$\int \frac{d^3 p}{(2\pi)^3}$, get

$$\frac{\partial N}{\partial t} + 3HN = -\Gamma(N - N_{eq})$$

Step # 2 change variables, $N = Y \cdot S$,

S: entropy density, $S = \frac{2\pi^2}{45} T^3 g_{eff}$

Entropy is conserved, $SR^3(t) = \text{Const}$
adiabatic expansion
equation for Y (Y $\qquad \dot{S} + 3HS = 0$

$$\dot{Y} S + \underbrace{Y\dot{S} + 3HY \cdot S}_{= 0} = -\Gamma(YS - Y_{eq} \cdot S)$$

$$\dot{Y} = -\Gamma(Y - Y_{eq}), \quad \Gamma = \Gamma(T) \quad (**)$$

$$Y_{eq} = \begin{cases} \text{Const}, & T \gg m \\ \frac{45 g}{8\pi^2 g_{eff}} \left(\frac{m}{2\pi T}\right)^{3/2} \exp\left(-\frac{m}{T}\right), & m \ll T \end{cases}$$

$$\text{Const} = \begin{cases} \frac{45 g\, \zeta(3)}{8\pi^4 g_{eff}}, & \text{bosons} \\ \frac{45 \cdot 3 \cdot g\, \zeta(3)}{8\pi^4 g_{eff}}, & \text{fermions} \end{cases}$$

Now, replace t by temperature,

as $t = \frac{M_0}{2T^2}$; $dt = -\frac{1}{H}\frac{dT}{T} \Rightarrow$

$$T\frac{dY}{dT} = + \frac{\Gamma}{H}(Y - Y_{eq})$$

from here one can see that the ratio $\frac{\Gamma}{H}$

is important for consideration

to simplify further the computation,

introduce $x = -\log\frac{T}{T_0}$; $dx = \frac{dT}{T} \Rightarrow$

$$\frac{dY}{dx} = -\frac{\Gamma}{H}(Y - Y_{eq})$$

Solution:

$$Y = C_0 \exp\left[-\int_{x_0}^{x}\frac{\Gamma}{H}dx\right] + Y_{eq}(x)$$

$$-\int_{x_0}^{x}\frac{dY_{eq}}{dx'}dx' \cdot \exp\left[-\int_{x'}^{x}\frac{\Gamma}{H}dx''\right]$$

$$\int_{t_0}^{t} \Gamma(t')dt' = -\int_{t_0}^{t} \frac{\Gamma}{H} \frac{dT}{T}$$

example: $e^+e^- \to \gamma\gamma$, $T \gg m_e \Rightarrow$

$$\frac{\Gamma}{H} = \frac{\alpha^2 \zeta(3)}{6\pi} \frac{T M_0}{T^2} \; ; \; \int \frac{\Gamma}{H} \frac{dT}{T} =$$

$$= \frac{\alpha^2 \zeta(3)}{6\pi} M_0 \int_{T_0}^{T} \frac{dT}{T^2} \approx \frac{\alpha^2 \zeta(3) M_0}{6\pi} \left(\frac{1}{T} - \frac{1}{T_0} \right)$$

may send to ∞

$$= \frac{\alpha^2 \zeta(3) M_0}{6\pi} \frac{1}{T}$$

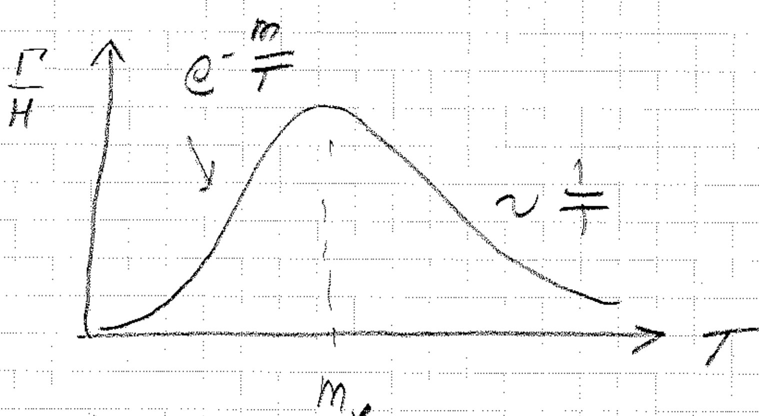
$$\int_0^t \Gamma(t')dt' \approx \frac{(10^{-2})^2 \cdot 10^{18}}{20} \frac{1}{T} \approx \frac{10^{13}}{T}$$

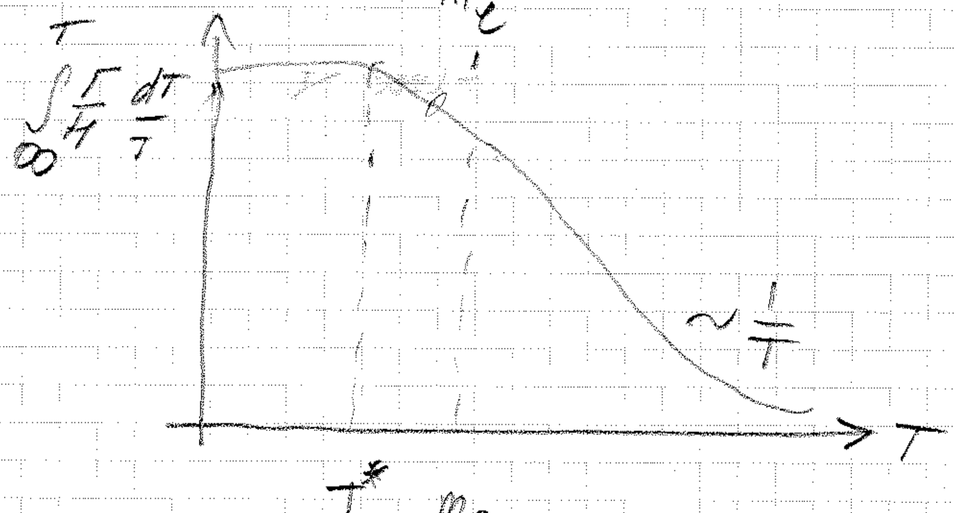
huge #: system is in th. eq at $T \gtrsim m_e$.

- first term in the solution: $C_0 \times \exp(\dots)$, extremely small
- third term in the solution: also small, since $\frac{dY_{eq}}{dt'} \approx 0$, and because of exp. suppression

what happens when T is $\lesssim m$

$$\frac{\Gamma}{H} \approx \pi r_e^2 \left(\frac{mT}{2\pi}\right)^{3/2} e^{-\frac{m}{T}} \cdot \frac{M_0}{T^2}$$





for $T \lesssim T^*$ integral is approximately constant:

To find T^*:

$$\int_{T^*}^{0} \frac{\Gamma}{H} \frac{dT}{T} \simeq 1 \quad (e\text{-times})$$

Γ is a very steep function of $T \Rightarrow$

$$\int_{T^*}^{0} \frac{\Gamma}{H} \frac{dT}{T} \propto \frac{\Gamma(T^*)}{H(T^*)} \Rightarrow$$

T^* is found from $\Gamma(T^*) = H(T^*)$

"freezing condition"!

Suppose $T > T^* \Rightarrow \quad \Gamma \gg H \Rightarrow$

we can write

$$Y = Y_{eq} - \frac{H}{\Gamma}\frac{dY}{dx} \Rightarrow$$

$$Y \approx Y_{eq} - \frac{H}{\Gamma}\frac{dY_{eq}}{dx}$$ — small deviation from thermal equilibrium

Suppose $T < T^*$, $\frac{\Gamma}{H} \lesssim 1 \Rightarrow$

$$Y \approx Y_{eq}(x) - \int_{x^*}^{x} \frac{dY_{eq}}{dx'}dx' \underbrace{\exp\{\ldots\}}_{\approx 1} \approx$$

$$Y_{eq}(x) - Y_{eq}(x) + Y_{eq}(x^*) -$$

QED

Conclusion

if $\Gamma > H$: concentration is equilibrium one (neglect "$H \frac{\partial n}{\partial p} \vec{p}$"),

$\Gamma = H$ freezing moment

$\Gamma < H$: particles do non interact (neglect $\Gamma(n - n_{eq})$)

approximate rules, work within factor of few!

Decoupling of photons

Saha's equilibrium formula:

$$\frac{n_e n_p}{n_H} = \left(\frac{m_e T}{2\pi}\right)^{3/2} e^{-I/T} \qquad e^- + p \rightleftharpoons H + \gamma$$

$I = 13.6\,eV = 1.58 \cdot 10^5\,{}^\circ K$ – ionization energy

$\sigma_{\gamma e} \approx \frac{8\pi}{3} r_e^2$ (Compton scattering)

$r_e = \frac{\alpha}{m_e}$ – electron radius

$\sigma_{\gamma p} \approx \frac{8\pi}{3} (r_p)^2 \ll \sigma_{\gamma e}$

$r_p \approx \frac{\alpha}{m_p}$

Go to page 29 if believe

Derivation of Saha's equation

hypothesis: $n_H \gg n_p$

Let we have plasma with

p ; e^- ; γ ; H

proton electron photon hydrogen

$$n_p = \left(\frac{m_p T}{2\pi}\right)^{3/2} g_p e^{-\frac{(m_p - \mu_p)}{T}}$$

g – spin factor

$$n_H = \left(\frac{m_H T}{2\pi}\right)^{3/2} g_H e^{-\frac{m_H - \mu_H}{T}}$$

equilibrium concentrations

$g_H = 4$

$(4 = 2 \times 2)$

electron proton

$$n_e = \left(\frac{m_e T}{2\pi}\right)^{3/2} g_e e^{-\frac{m_e - \mu_e}{T}}$$

$n_\gamma =$ ... ultrarel. equation

chemical equilibrium: $\mu_p + \mu_e = \mu_H$

(as chemical potential of photon $= 0$)

$$\frac{n_p \cdot n_e}{n_H} = \left(\frac{m_e T}{2\pi}\right)^{3/2} e^{-\frac{m_p - m_e + m_H}{T}} =$$

$$= \left(\frac{m_e T}{2\pi}\right)^{3/2} e^{-I/T}$$

QED

$n_e = n_p$ (neutrality of plasma)

$$n_e \approx n_H^{1/2} \left(\frac{m_e T}{2\pi}\right)^{3/4} e^{-\frac{I}{2T}}$$

$$n_H \approx \frac{n_B}{n_\gamma} \cdot n_\gamma \approx \eta \cdot n_\gamma \quad ; \quad \eta = (6.15 \cdot) \cdot 10^{-10}$$

Photon interaction rate:

$$\Gamma = \sigma n_e v \approx \frac{8\pi\alpha^2}{3m_e^2} \cdot \eta^{1/2} \left[\frac{\zeta(3)}{\pi^2} \cdot 2T^3\right]^{1/2} \cdot$$

$$\cdot \left(\frac{m_e T}{2\pi}\right)^{3/4} e^{-I/2T} = \frac{T^2}{M_0}$$

$$M_0 = \frac{M_{pl}}{1.66 N_{eff}^{1/2}} \approx \frac{1.2 \cdot 10^{19}}{1.66 \cdot \sqrt{2}} \approx 5 \cdot 10^{18} \text{ GeV}$$

$$e^{-\frac{I}{2T}} \underbrace{\frac{8 \cdot \pi \cdot \alpha^2}{3} \cdot \eta^{1/2} \cdot \left(\frac{2\zeta(3)}{\pi^2}\right)^{1/2} \left(\frac{1}{2\pi}\right)^{3/4}}_{\text{Number ,,B''}} \cdot \frac{T^{3/4}}{m_e^{5/4}} = \frac{T^2}{M_0}$$

$$e^{-\frac{I}{2T}} \cdot B \frac{2^{1/4} T^{1/4}}{I^{1/4}} = \frac{2^{1/4} m_e^{5/4} \cdot 2}{I^{1/4} M_0} \qquad \frac{I}{2T} = x$$

$$\frac{1}{x^{1/4}} e^{-x} = \frac{1}{B} \frac{2^{1/4} m_e^{5/4}}{I^{1/4} M_0} = 1.24 \cdot 10^{-12} \,!$$

$$x = 26.6 \Rightarrow$$

$$T = \frac{I}{26.6 \cdot 2} = 3000°K = 0.25 \text{ eV}$$

Conclusion:

Recombination occures at $\approx 3000\,^{\circ}K$

Below $3000\,^{\circ}K$ photons do not interact with plasma. Photons "decouple" at $T \approx 3000\,^{\circ}K$

Age of the Universe at that time:

$$t = 0.301\, g_*^{-1/2} \frac{M_{pl}}{T^2} = 0.301 \frac{1}{\sqrt{2}} \frac{1.2\cdot 10^{19}}{(0.25\cdot 10^{-9})^2} = 2.7\cdot 10^{13}\,s \approx 10^{5}\ \text{years}$$

Red shift: $z = \dfrac{\omega_{emit}}{\omega_{rec}} - 1 \approx \dfrac{T_{emit}}{T_{rec}} - 1 \approx 1000$

Derivation of recombination is not exact, as we assumed that $\sigma_{\gamma H}$ (γ on neutral hydrogen) is equal to zero.

Also, we considered only ground state of H atom. But result is actually quite precise.

Interesting: temperature of recombination is more or less the same (factor 3) as temperature of transition from radiation dominated universe to matter dominated Universe:

~~(see page 16)~~

$z_{equality} \simeq 3000$

$z_{decoupling} \simeq 1100$