main points of #6.

discussed kinetic equations and freezing approximation:

\[ \Gamma(t^*) \mathcal{H}(t^*) = 1 \implies \]

\[ N = \begin{cases} 
\mathcal{N} \mathcal{G}, & \text{if } \Gamma(t) > \mathcal{H}(t) \\
\frac{\mathcal{N} \mathcal{G} (t^*) (\mathcal{K}(t^*) )^2}{\mathcal{K}(t)}, & \text{if } \Gamma(t) < \mathcal{H}(t),
\end{cases} \]

and particle is stable

\[ \Gamma \] only takes into account reactions that change \( N \)

Today:

- entropy conservation

- demonstration that we have indeed this solution

- decoupling of photons
Entropy conservation in expanding Universe

For simplicity, closed universe.

Let $S$ be the total entropy.

Consider expansion of $\frac{1}{S} \frac{dS}{dt}$ with respect to

$$H = \frac{1}{a} \frac{da}{dt}.$$ 

$$\frac{dS}{dt} = A + BH + CH^2 + \ldots$$

$A = 0$, since if $H = 0$, then $\frac{dS}{dt} = 0$

(we are already in the state of thermal equilibrium).

$B = 0$, since $\frac{dS}{dH} > 0$, but we can change
the sign by $H$

$$\Rightarrow \quad \frac{dS}{dt} = CH^2 \quad \text{from dimensional reasons}.$$ 

$c = s \Rightarrow \quad c \approx t_0$ - reaction time $\Rightarrow$

$$\frac{1}{S} \frac{dS}{dt} = t_0 H^2 \Rightarrow \quad \frac{8S}{3} = \int_{t_0}^{t} (H(t)) H^2(t) dt$$

$$= 0 \int_{t_0}^{t} \gamma(t) dt$$

$$= \frac{1}{1 + \frac{a(t)}{a(t_0)}}$$
Solution to kinetic equation

Step #1: consider integrated version of (1) - (momentum dependence is not very interesting)

\[ \int \frac{d^3 p}{(2\pi)^3}, \text{ set} \]

\[ \frac{dN}{dt} + 3N V = -\Gamma (N - N_{eq}) \]

Step #2: change variables, \( N = Y \cdot S \)

\( S \): entropy density, \( S = \frac{2\pi^2 T^3}{45} \text{ sed} \)

Entropy is conserved, \( 3R^3(t) = \text{Const} \)

Adiabatic expansion equation for \( Y \)

\( \dot{S} + 3HS = 0 \)

\( \dot{Y} S + Y \dot{S} + 3HY S = -\Gamma (YS - Y_{eq} S) \)

\( \Gamma(Y - Y_{eq}) \)

\( \Gamma = \Gamma(T) \) \( (x+1) \)

\( Y_{eq} = \left\{ \begin{array}{ll}
\text{Const, } T \gg m \\
\frac{45 \pi}{2 \pi T} \left( \frac{m}{2\pi T} \right)^{3/2} \exp \left( -\frac{m^2}{2T} \right), \ m \ll T
\end{array} \right. \)

\( \text{Const} = \left\{ \begin{array}{ll}
\frac{45 \pi^{2/3}}{2 \pi T \text{ sed}} , \text{ bosons} \\
\frac{45 \pi^{2/3}}{8 \pi T \text{ sed}} , \text{ fermions}
\end{array} \right. \)
Now, replace $\tau$ by temperature, as

$$
t = \frac{M_0}{27^2}; \quad dt = -\frac{1}{H} \frac{dT}{T} \\
$$

$$
-\frac{dY}{dt} = \frac{\Pi}{H} (Y - Y_{eq})
$$

from here one can see that the ratio $\frac{\Pi}{H}$ is important for consideration.

to simplify further the computation, introduce $x = -\log \frac{\tau}{\tau_0}$; 

$$
dx = \frac{dT}{T} \\
$$

$$
\frac{dY}{dx} = -\frac{\Pi}{H} (Y - Y_{eq})
$$

Solution:

$$
Y = C_0 \exp \left[ -\int_{x_0}^{x} \frac{\Pi}{H} \, dx \right] + Y_{eq}(x)
$$

$$
- \int_{x_0}^{x} \frac{dY_{eq}}{dx'} \, dx' \cdot \exp \left[ -\int_{x'}^{x} \frac{\Pi}{H} \, dx'' \right]
$$
\[
\int_{t_0}^{t} \Gamma(t) dt' = - \int_{t_0}^{t} \frac{1}{H} \frac{dT}{T}
\]

Example: \(e^t e^{-t} \to 0\), \(T \to \infty \Rightarrow \)

\[
\frac{\Gamma}{H} = \frac{2\pi \hbar}{6 \pi} \frac{T^2 M_0}{T^2} \Rightarrow \int_{t_0}^{T} \frac{\Gamma}{H} \frac{dT}{T} =
\]

\[
= \frac{2\pi \hbar}{6 \pi} \frac{M_0}{T_0} \int_{t_0}^{T} \frac{dT}{T^2} \approx \frac{2\pi \hbar}{6 \pi} \frac{M_0}{T_0} \left( \frac{1}{T} - \frac{1}{T_0} \right)
\]

\[
= \frac{2\pi \hbar}{6 \pi} \frac{M_0}{T_0} \frac{1}{T} \to \infty \text{ as } T \to \infty
\]

\[
\int_{t_0}^{T} \rho(11) dt' \approx \frac{(10^{-2})^2 \cdot 10^{13}}{20} \frac{1}{T} \approx \frac{10^{13}}{T}
\]

huge \#; system is in the 8

at \(T \to \infty\)

---

first term in the solution: \(e \times \exp(\cdots)\),

extremely small

---

third term in the solution: also small,

since \(\frac{d\gamma_0}{dt'} \approx 0\), and because of exp.

suppression.
what happens when \( T \leq T^* \)

\[
\frac{\Gamma}{H} \approx \pi \hbar^2 \left( \frac{mT}{2\pi} \right)^{3/2} e^{-\frac{m}{T}} \cdot \frac{M_0}{T^2}
\]

\[
\frac{\Gamma}{H} \sim \frac{1}{T}
\]

\[
\frac{\Gamma}{H} \sim \frac{1}{T}
\]

For \( T \leq T^* \) integral is approximately constant:

\[
\int_0^{T^*} \frac{\Gamma}{H} \, dt \sim 1 \quad (e^{-t} \text{ times})
\]

\( T^* \) is a very steep function of \( T \) \( \Rightarrow \)

\[
\int_0^{T^*} \frac{\Gamma}{H} \, dt \propto \frac{\Gamma(T^*)}{\hbar(T^*)} \Rightarrow
\]

\( T^* \) is found from \( \Gamma(T^*) = \hbar(T^*) \)

"freezing condition" \( \)
Suppose \( T > T^* \Rightarrow \frac{\Gamma}{H} \Rightarrow \)

we can write

\[
Y = Y_{eq} - \frac{H}{\Gamma} \frac{dy}{dx} \Rightarrow
\]

\[
Y \approx Y_{eq} - \frac{H}{\Gamma} \frac{dy}{dx} - \text{small deviation from thermal equilibrium}
\]

Suppose \( T < T^* \), \( \frac{\Gamma}{H} \ll 1 \Rightarrow \)

\[
Y \approx Y_{eq}(x) - \int_{x^*}^{x} \frac{dy}{dx} \, dx' \exp\left\{-\frac{\beta}{\Gamma} \right\} \Rightarrow \]

\[
Y \approx \frac{1}{2} \left( Y_{eq}(x) - Y_{eq}(x^*) + Y_{eq}(x^*) \right) - \text{QED}
\]
Conclusion

If \( \Gamma > H \): concentration as equilibrium one (neglect \( \frac{H^2}{\eta} \)),

\[ \Gamma = H : \text{freezing moment} \]

\[ \Gamma < H : \text{particles do not interact} \]

(approximate rules, work within factor of few!)

Decoupling of photons

Saha's equilibrium formula:

\[
\frac{n_e n_p}{n_H} = \left( \frac{m_e T}{2 \pi} \right)^{3/2} e^{-\frac{I}{T}} \quad e^+p \approx H+Y
\]

\[ I = 13.6 \text{eV} = 1.58 \times 10^5 \text{K} \quad \text{ionization energy} \]

\[ \sigma_{ge} \approx \frac{8\pi}{3} r_e^2 \quad \text{(Compton scattering)} \]

\[ r_e = \frac{e^2}{m_e} \quad \text{electron radius} \]

\[ \sigma_{gp} \approx \frac{8\pi}{3} (r_p)^2 \ll \sigma_{ge} \]

\[ r_p = \frac{e^2}{m_p} \]

Go to page 29 if believe
Derivation of Saha's equation

Let we have plasma with

\[ n_p = \left( \frac{m_p T^{3/2}}{2\pi} \right)^{3/2} \rho_p \ e^{-\frac{m_p - \mu_p}{kT}} \] 

\[ n_H = \left( \frac{m_H T^{3/2}}{2\pi} \right)^{3/2} \rho_H \ e^{-\frac{m_H - \mu_H}{kT}} \] 

spin factor

\[ n_e = \left( \frac{m_e T^{3/2}}{2\pi} \right)^{3/2} \rho_e \ e^{-\frac{m_e - \mu_e}{kT}} \] 

equilibrium concentrations

\[ n_p \cdot n_e = \left( \frac{m_e T^{3/2}}{2\pi} \right)^{3/2} \ e^{-\frac{m_e - \mu_e}{kT}} \]

\[ = \left( \frac{m_e T^{3/2}}{2\pi} \right)^{3/2} \ e^{-\frac{m_e - \mu_e}{kT}} \]

\[ \]
\[ n_e = n_p \quad (\text{neutrality of plasma}) \]

\[ n_e \approx n_H \left(\frac{e_{H}}{2\pi}\right)^{3/4} e^{-\frac{T}{2T_H}} \]

\[ n_H \approx n_B, \quad n_B \approx \eta \cdot n_P \quad ; \quad \eta = \left(\frac{6.15}{2.7}\right) \cdot 10^{-10} \]

Photon interaction rate:

\[ \Gamma = 6 \cdot n_e \nu \approx \frac{8\pi^2}{3m_e^2} \cdot \eta \cdot \left[\frac{\varepsilon(3)}{\pi^2} \cdot 2T^3\right]^{1/2} \]

\[ \left(\frac{m_e T}{2\pi}\right)^{3/4} e^{-\frac{T}{2T_H}} = \frac{T^2}{M_0} \]

\[ M_0 = \frac{\frac{M_0}{1.66 \times 10^{-19}}}{1.66 \times 12} \approx 5 \cdot 10^{18} \text{ GeV} \]

\[ e^{-\frac{T}{2T_H}} \left(\frac{8\pi^2}{3} \cdot \frac{\nu}{\gamma} \right)^{1/2} \cdot \left(\frac{2513}{\pi^2}\right)^{1/2} \cdot \left(2\pi\right)^{3/4} \cdot \frac{T^{3/4}}{m_e^{3/4}} \cdot \frac{T^2}{M_0} = \frac{T^2}{M_0} \]

Number \( n_B \)

\[ e^{-\frac{T}{2T_H}} \cdot B \left(\frac{T^{3/4}}{M_0}\right) \left(\frac{T^{1/4}}{I^{1/4}}\right) = \frac{8^{1/4} m_e^{3/4} \cdot 2}{I^{1/4} M_0} \cdot \frac{T^{3/4}}{M_0} = \frac{T^2}{M_0} \]

\[ \frac{1}{x^{1/4}} e^{-x} = \frac{\Gamma}{B} \left(\frac{T^{1/4}}{I^{1/4}}\right) \left(\frac{2.74 \cdot 10^{-12}}{I^{1/4} M_0} \right) = 1.24 \cdot 10^{-12} \cdot \]

\[ x = 26.6 \Rightarrow \]

\[ T = \frac{x}{26.6} = 3000^\circ K = 0.25605 \]
Conclusion:
Recombination occurs at $T = 3000^\circ K$.
Below $3000^\circ K$ photons do not interact with plasma. Photons "decouple" at $T = 3000^\circ K$.

Age of the Universe at that time:

$$t = 0.301 \, g^{1/2} \, \frac{Mpc}{T^2} = 0.301 \, \frac{1}{\sqrt{2}} \, \frac{1.2 \times 10^{18}}{(0.25 \times 10^{-3})^2} = 2.7 \times 10^8 \, \text{s} 
\approx 10^3 \, \text{years}$$

Red shift:
$$z = \frac{\text{wavelength}}{\text{wavelength}} - 1 = \frac{T_{\text{emitted}}}{T_{\text{rec}}} - 1 \approx 1000$$

Derivation of recombination is not exact, as we assumed that $\sigma_H$ (cross section on neutral hydrogen) is equal to zero.

Also, we considered only ground state of H atom. But result is actually quite precise.

Interesting: temperature of recombination is more or less the same as temperature of transition from radiation dominated universe to matter dominated Universe:

$$z = \text{equality} \approx 3000$$
$$z = \text{decoupling} \approx 1100$$