Relativity and cosmology II

Lecture #6

Main points & #5

- Considered foundations for Big Bang Theory.
  
  Initial density in the past $\rightarrow$ equilibrium $\rightarrow$ everything is described by temperature.

\[
\frac{4\pi}{30} \rho T^4 = \frac{3}{32\pi G t^2} \quad \Rightarrow
\]

Relation between $T$ and $t$:

\[
R \sim t^{1/2}, \quad H = \frac{1}{2} \frac{1}{t} \Rightarrow H = \left(\frac{\frac{3}{5} \rho T^4 \cdot \frac{32\pi G}{3} \frac{1}{t^2}}{4}\right)^{1/2}
\]

Today: Big bang continued

- Some extra formulas:
  - Kinetic description,
    Boltzmann equation
  - Relaxation time approximation
  - Solution to kinetic equation

\[
\rho_0 = \left(\frac{45}{45^3 \frac{g \cdot s^2}{G}}\right)^{1/2} = \frac{m_p}{1.66 \times 10^{-12}}
\]
Particle kinetics in expanding universe.

Homogeneous case:

\[ n = n(0, t). \]

Suppose: \( n(0, t) = n_0(\frac{\mathbf{p}}{a(t)}) \)

No interactions:

\( p \cdot a(t) = \text{const} \) (red shift)

So:

\[ n(\mathbf{p}, t) = n_0 \left( \frac{\mathbf{p} \cdot a(t)}{a(t_0)} \right) = n_0 \left( \frac{\mathbf{p} \cdot a(t)}{a(t_0)} \right) \]

Nothing happens with the particles, just change of their momentum.

Kinetic equation:

\[ \frac{\partial n}{\partial t} = \frac{\partial n_0}{\partial \mathbf{x}} \cdot \frac{\partial \mathbf{v}}{\partial t} - \frac{\partial n_0}{\partial \mathbf{v}} \cdot \frac{\partial \mathbf{v}}{\partial t} + \frac{\partial n_0}{\partial \mathbf{p}} \cdot \frac{\partial \mathbf{v}}{\partial \mathbf{p}} \]

\[ \frac{\partial n}{\partial \mathbf{p}} = \frac{\partial n_0}{\partial \mathbf{v}} \cdot a(t) \]

\[ \Rightarrow \frac{\partial n_0}{\partial \mathbf{v}} = \frac{a(t)}{a(t_0)} \frac{\partial n}{\partial \mathbf{p}} \]

\[ \frac{\partial n}{\partial t} - \mathbf{p} \cdot \frac{\partial n}{\partial \mathbf{p}} = 0 \]

for number density:

\[ N = \int d^3 \mathbf{p} \ n(\mathbf{p}) \]
\[ \frac{dn}{dt} - \sum p_i \frac{\partial n}{\partial p_i} \, d^3p = 0 \]

\[ \sum p_i \frac{\partial n}{\partial p_i} \, d^3p = - \sum \frac{\partial n}{\partial \lambda} \cdot \mathbf{n} \cdot d^3p \Rightarrow \]

\[ \frac{dn}{dt} + 3Hn = 0 \Rightarrow \# \text{Nukerons as } n \sim \frac{1}{q^3} \]

Taking into account collisions:

\[ \frac{dn}{dt} - \mu p \frac{\partial n}{\partial p} = J_{\text{coll}} \]

\( J_{\text{coll}} \): collision integral, determined by probabilities of reactions

\[ n \rightarrow q_1 \]
\[ q_1 \rightarrow q_2 \]
\[ q_1 \rightarrow q_3 \]

\[ J_{\text{coll}} = \frac{1}{2p_0} \int \frac{d^3q_1}{(2\pi)^3 2q_0} \frac{d^3q_2}{(2\pi)^3 2q_0} \frac{d^3q_3}{(2\pi)^3 2q_0} \]
\[ \times (2\pi)^4 \delta(p - q_1 - q_2 - q_3) |M_{ji}|^2 \]
\[ \sum n(p) \, n(q_1) (1 \pm n(q_2)) (1 \pm n(q_3)) - \]
\[ \rightarrow \text{final nukerons} \]
\[ n(q_3) n(q_2) (1 \pm n(q)) (1 \pm n(p)) \]
Kinetic equations are complicated

"Reasonable approximation"

relaxation:

$$\frac{2n}{\delta t} - \nabla \cdot \frac{2n}{\delta \vec{r}} = -\Gamma (n - n_e)$$  \hspace{1cm} (x)

"rate" of reaction

Correct solution: \( n \approx n_e \) if no expansion

What is \( \Gamma \)?

related to typical mean free time

let \( \sigma \): cross-section, "area of the particle"

\( \sigma \): \( \text{cm}^2 = \text{GW}^{-2} \)

mean free path:

\( \lambda = \frac{1}{\sigma n} \rightarrow n - \text{concentration} \)

mean free time:

\( \tau_x = \frac{2}{\lambda v} = \frac{1}{\sigma n v} \)

\( \tau_x \approx \bar{\lambda} = \langle \sigma n v \rangle \)
Every simpler equation can be considered.

for total particle density

\[ N = \int \frac{d^3p}{(2\pi)^3} \, n(p) \]

integrate

\[ \frac{\partial n}{\partial t} - \mathbf{p} \cdot \frac{\partial n}{\partial \mathbf{p}} = -\Gamma (N - N_{eq}) \left| \times \int \frac{d^3p}{(2\pi)^3} \right| \]

right hand side:

\[ \rightarrow -\Gamma (N - N_{eq}) \]

left hand side:

\[ \int \frac{\partial n}{\partial t} \frac{d^3p}{(2\pi)^3} \rightarrow \frac{\partial N}{\partial t} \]

\[ \int \mathbf{p} \cdot \frac{\partial n}{\partial \mathbf{p}} \frac{d^3p}{(2\pi)^3} \rightarrow \int \frac{\partial \mathbf{p}}{\partial \mathbf{p}} \cdot \mathbf{n} \frac{d^3p}{(2\pi)^3} = -3N(t) \]

integration

by parts, \( n(p) \rightarrow 0 \) at \( p \rightarrow \infty \)

\[ \frac{\partial N}{\partial t} + 3\chi N = -\Gamma (N - N_{eq}) \]
Solution to kinetic equation:

Find temperature \( T^* \) at which

\[ \Gamma(T^*) = H(T^*) \]

Then

\[ N_{\Gamma} \begin{cases} \frac{M_{\Gamma}}{K} & \text{if } \Gamma(T) > H(T); \quad T > T^* \\ \frac{M_{\Gamma}(T^*)}{K} \left( \frac{R(T^*)}{R(T)} \right)^3 & \text{if } \Gamma(T) < H(T); \quad T < T^* \end{cases} \]

\( T^* \): freezing moment

Important remarks:

- To compute \( \Gamma \) in (\( *) \) one has to take into account only reactions which change the number of particles under consideration.

- Equation (\( ** \)) is not true for non-stable particles. The decays should be considered separately.
Example

\[ e^+e^- \rightarrow \gamma \gamma \quad (\text{center of mass}) \]

\[ \sigma \approx \begin{cases} 
\frac{1}{\pi \epsilon^2} \sum \epsilon_k^2, & \epsilon \ll 1 \\
\frac{m^2}{\epsilon^2} \sum \epsilon_k^2 \left( \log \frac{4 \epsilon_k^2}{m^2} - 1 \right), & \epsilon \approx 1 
\end{cases} \]

\[ \epsilon_k = \frac{m}{m_0}, \quad \epsilon \approx \frac{1}{137} \]

\[ \Gamma \approx \text{const} \epsilon^2 \frac{m}{\epsilon^2} \begin{cases} 
\frac{\sum \epsilon_k^2}{\epsilon^2}, & 0 < T \ll m \\
\frac{T \epsilon_k^2}{\epsilon^2}, & \frac{3}{2} \frac{\epsilon_k^2}{m^2} T^2 \approx 1, \quad T \approx m \\
\epsilon_k^2 \sim 3T \end{cases} \]

\[ \Gamma \sim e^{-m/T} \quad T \propto T \]

\[ \Gamma \rightarrow m \text{ at high } T, \quad \text{and small at small } T \]

"Normal" situation: rate is large at high T, and small at small T.

Exactly what one could expect.

Plasma is dense \Rightarrow \text{ particles interact often.}"
constant equal to equilibrium concentration at $T^*$

**Example:**

For electrons: equation for $T^*$:

$$\pi \rho e^2 \left( \frac{mT^*}{2\pi} \right)^{3/2} e^{-\frac{m}{T^*}} - \frac{m}{T^*} = 1$$

$$\alpha = \frac{m}{T^*} \Rightarrow T^* = \frac{m}{\alpha} \Rightarrow$$

$$\pi \rho e^2 \left( \frac{m^2}{2\pi} \right)^{3/2} e^{-\alpha} \cdot \frac{M_0}{m^2} x^2 = 1$$

$$\alpha^{3/2} e^{-\alpha} = \frac{1}{\pi \rho e^2} \left( \frac{2\pi}{3} \right)^{3/2} \frac{1}{M_0 m} \approx$$

$$\approx \frac{2 \sqrt{2\pi}}{M_0 m} \approx 10^{-4} \frac{10^{-3}}{10^{-12}} \approx$$

$$\approx 10^{-17}$$

$$\frac{1}{2} \log \frac{\alpha}{\alpha} = -3.9 \Rightarrow$$

$$\alpha \approx 39 + \frac{1}{2} \log 39 \approx 43$$

$$So: \frac{m}{T^*} = 43; \quad \frac{n_e}{n_0} | T = T^* \approx \left( \frac{43}{6\pi} \right)^{3/2} e^{-\frac{43}{2} \frac{1.5}{2}} \approx 3.0 \times 10^{12}$$