Relativity and cosmology II

Lecture # 6

Main points of #5

- considered foundations for Big Bang Theory:
  high density in the past ⇒ equilibrium → everything is described by temperature

$$\frac{\pi^2}{30} g_* T^4 = \frac{3}{32\pi G t^2} \Rightarrow$$

relation between T and t:

$R \sim t^{1/2}$; $H = 1/2t \Rightarrow H = \left(\frac{\pi^2}{30} g_* T^4 \cdot \frac{32\pi G}{3} \frac{1}{4}\right)^{1/2} = \frac{T^2}{M_0}$

Today: Big Bang continued

- Some extra formulas
- kinetic description, Boltzmann equation
- relaxation time approximation
- Solution to kinetic equation

$$M_0 = \left(\frac{45}{4\pi^3 g_* G}\right)^{1/2} = \frac{M_P}{1.66\, g_*^{1/2}}$$

particle kinetics in expanding universe.

Homogeneous case:

$n = n(p,t)$. Suppose: $n(p,t_0) = n_0(\vec{p}) \equiv n_0(\vec{x})$

No interactions:

$p \cdot a(t) = \text{const}$ (red shift)

So: $n(\vec{p},t) = n\left(\frac{\vec{p}\cdot a(t)}{a(t_0)}, t_0\right) = n_0\left(\frac{\vec{p}\,a(t)}{a(t_0)}\right)$

nothing happens with the particles, just change of their momentum

Kinetic equation:

$$\frac{\partial n}{\partial t} = \frac{\partial n_0}{\partial \vec{x}} \frac{\partial \vec{x}}{\partial t} = \frac{\partial n_0}{\partial \vec{x}} \frac{\vec{p}\,\dot{a}(t)}{a(t_0)}$$

$$\frac{\partial n}{\partial \vec{p}} = \frac{\partial n_0}{\partial \vec{x}} \frac{a(t)}{a(t_0)} \Rightarrow \frac{\partial n_0}{\partial \vec{x}} = \frac{a(t_0)}{a(t)} \frac{\partial n}{\partial \vec{p}} \Rightarrow$$

$$\boxed{\frac{\partial n}{\partial t} - H\vec{p}\,\frac{\partial n}{\partial \vec{p}} = 0}$$

for number-density:

$$N = \int \frac{d^3p}{(2\pi)^3}\, n(p):$$

$$\frac{dN}{dt} - \int H \vec{p}\, \frac{\partial n}{\partial \vec{p}}\, d^3p = 0$$

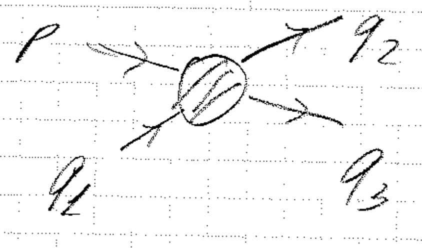
$$\int p_x \frac{\partial n}{\partial p_x}\, d^3p = -\int \frac{\partial p_x}{\partial p_x} \cdot n \cdot d^3p \Rightarrow$$

$$\frac{dN}{dt} + 3HN = 0 \Rightarrow \# N \text{ decreases as} \sim \frac{1}{a^3}$$

Taking into account collisions:

$$\frac{\partial n}{\partial t} - H \vec{p}\, \frac{\partial n}{\partial p} = I_{coll}$$

I_{coll}: collision integral, determined by probabilities of reactions



$$I_{coll} = \frac{1}{2p^0} \int \frac{d^3q_1}{(2\pi)^3 2q_1^0}\, \frac{d^3q_2}{(2\pi)^3 2q_2^0}\, \frac{d^3q_3}{(2\pi)^3 2q_3^0} \cdot$$

$$\cdot (2\pi)^4 \delta^4(p+q_1-q_2-q_3) \cdot |M_{fi}|^2$$

$$\cdot \left[n(p)\, n(q_1)\, (1 \pm n(q_2))(1 \pm n(q_3)) - \right.$$

initial → final particles

$$\left. - n(q_3)\, n(q_2)\, (1 \pm n(p))(1 \pm n(q_1)) \right]$$

Kinetic equations are complicated (4)

"Reasonable approximation"

relaxation :

$$\frac{\partial n}{\partial t} - H\vec{p}\,\frac{\partial n}{\partial p} = -\Gamma\,(n - n_{eq}) \qquad (*)$$

"rate" of reaction

Correct solution: $n \to n_{eq}$ if no expansion

What is Γ?

related to typical mean free time:

let σ: cross-section, "area of the particle" σ : $cm^2 = GeV^{-2}$

mean free pass:

$$\lambda = \frac{1}{\sigma n} \qquad n\text{ - concentration}$$

mean free time: $t^* = \frac{\lambda}{v} =$

$$= \frac{1}{\sigma n v}$$

$$\Gamma \approx \frac{1}{t^*} = \langle \sigma n v \rangle$$

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Even simpler equation can be considered - for total particle density

$$N = \int \frac{d^3p}{(2\pi)^3} n(p)$$

integrate

$$\frac{\partial n}{\partial t} - H\vec{p}\frac{\partial n}{\partial \vec{p}} = -\Gamma(n - n_{eq}) \Big/ * \int \frac{d^3p}{(2\pi)^3}$$

right hand side:

$$\rightarrow -\Gamma(N - N_{eq})$$

Left hand side:

$$\int \frac{\partial n}{\partial t} \frac{d^3p}{(2\pi)^3} \rightarrow \frac{\partial N}{\partial t}$$

$$\int \vec{p}\frac{\partial n}{\partial \vec{p}} \cdot \frac{d^3p}{(2\pi)^3} \underbrace{\rightarrow}_{\substack{\text{integration} \\ \text{by parts, } n(p)\to 0 \text{ at } p\to\infty}} -\int \frac{\partial \vec{p}}{\partial \vec{p}} \cdot n \frac{d^3p}{(2\pi)^3} = -3N(t) \Rightarrow$$

$$\frac{\partial N}{\partial t} + 3HN = -\Gamma(N - N_{eq})$$

Solution to kinetic equation:

find temperature T^* at which
$\Gamma(T^*) = H(T^*)$

Then

$$N = \begin{cases} N_{eq}(T), & \text{if } \Gamma(T) > H(T);\ T > T^* \\ N_{eq}(T^*)\left(\frac{R(T^*)}{R(T)}\right)^3, & \text{if } \Gamma(T) < H(T),\ T < T^* \end{cases}$$

T^*: freezing moment

important remarks:

- to compute Γ in (*) one has to take into account only reactions which change the number of particles under consideration

- equation (**) is not true for non-stable particles. The decays should be considered separately

Example

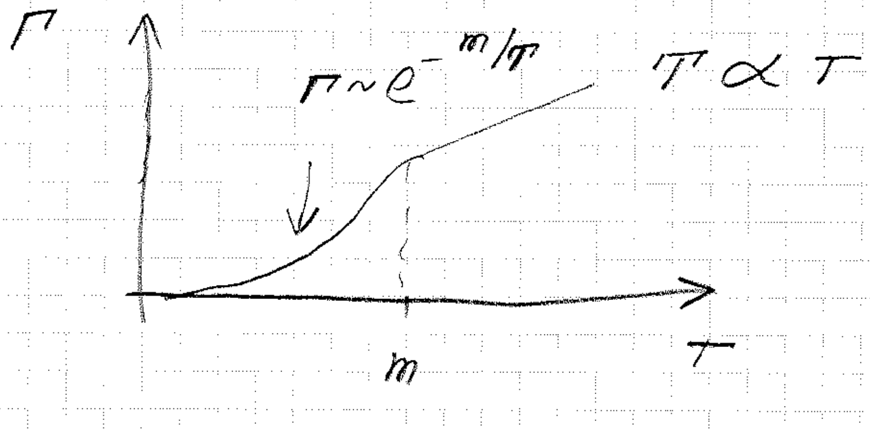
$e^+e^- \to \gamma\gamma$ (center of mass)

$$\sigma \approx \begin{cases} \frac{1}{2v} \pi r_e^2, & v \ll 1 \\ \frac{m^2}{\varepsilon^2} \pi r_e^2 \left(\log \frac{4\varepsilon^2}{m^2} - 1 \right), & v \approx 1 \end{cases}$$

$$r_e = \frac{\alpha}{m_e}; \quad \alpha \sim \frac{1}{137}$$

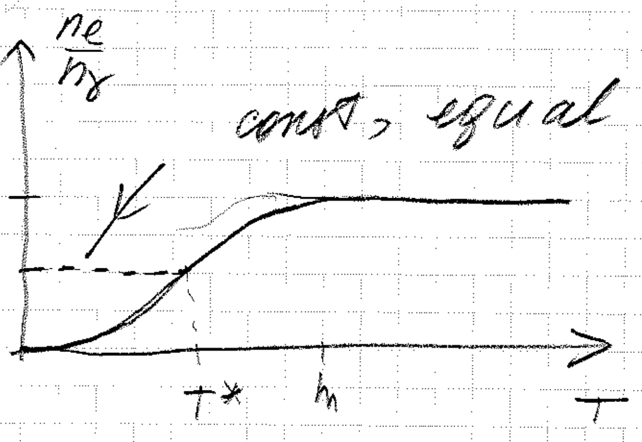
$$\Gamma \simeq \langle \sigma n v \rangle \simeq \begin{cases} \frac{\pi r_e^2}{2} \cdot 2 \cdot \left(\frac{mT}{2\pi} \right)^{3/2} e^{-\frac{m}{T}}, & T \lesssim m \\ \langle \frac{\pi \alpha^2}{\varepsilon^2} \rangle \cdot \frac{3}{2} \frac{\zeta(3)}{\pi^2} T^3, & T \gtrsim m \end{cases}$$

$\langle \varepsilon \rangle \simeq 3T$ $\quad \approx \pi \alpha^2 \cdot \frac{1}{6} \frac{\zeta(3)}{\pi^2} T$



"Normal" situation: rate is large at high T, and small at small T

exactly, what one could expect: plasma is dense ⇒ particles interact often



const, equal to equilibrium concentration at T^*

example

for electrons: equation for T^*:

$$\pi r_e^2 \left(\frac{mT^*}{2\pi}\right)^{3/2} e^{-\frac{m}{T^*}} \cdot \frac{M_0}{T_*^2} = 1$$

$$x = \frac{m}{T^*}; \quad T^* = \frac{m}{x} \Rightarrow$$

$$\pi r_e^2 \left(\frac{m^2}{2\pi x}\right)^{3/2} e^{-x} \cdot \frac{M_0}{m^2} x^2 = 1;$$

$$x^{1/2} e^{-x} = \frac{1}{\pi r_e^2} (2\pi)^{3/2} \frac{1}{M_0 m} \approx$$

$$\approx \frac{9\sqrt{2\pi}}{\alpha^2} \frac{m}{M_0} \approx 10^4 \frac{10^{-3}}{10^{18}} \approx$$

$$\approx 10^{-17}$$

$$\frac{1}{2}\log x - x = -39 \Rightarrow$$

$$x \approx 39 + \frac{1}{2}\log 39 \approx 43$$

So: $\frac{m}{T^*} = 43$; $\left.\frac{n_e}{n_\gamma}\right|_{T \le T^*} \approx \left(\frac{43}{\pi}\right)^{3/2} e^{-43} \cdot \frac{\pi^2}{\zeta(3)} \approx 3 \cdot 10^{-17}$