Relativity and cosmology II

Lecture # 5

Main points of # 4
- introduced critical density
- considered evolution of the Universe for $P = \lambda = 0$ and found that for $\rho \leq \rho_c$ Universe expands forever, and for $\rho < \rho_c$ collapses in the future

- found Hubble law (red shift - luminosity distance relation) for the general case: Universe containing radiation, matter & cosmological constant

General formular:

$$d(z) = \frac{1+z}{H_0 \Omega_k^{1/2}} \sinh\left[\Omega_k^{1/2} \int_{\frac{1}{1+z}}^{1} \frac{dx}{x^2 A(x)}\right]$$

where

$$A^2(x) = \Omega_\Lambda + \Omega_M \frac{1}{x^3} + \frac{\Omega_k}{x^2} + \frac{\Omega_\gamma}{x^4}$$

and $\Omega_k = 1 - (\Omega_\gamma + \Omega_M + \Omega_\Lambda)$

This f-la is valid for $\Omega_k < 0$ as well – complex „i" cancells away

Today:

- Summary of different behaviours
- Foundations of Big Bang theory

Cosmological constant domination

assume $\rho = 0$, $p = 0$, $k = 0$, but $\Lambda \neq 0$

$$\frac{\dot{R}^2}{R^2} = \frac{\Lambda}{3} \qquad [\text{valid only if } \Lambda > 0]$$

$$2\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} - \Lambda = 0$$

Solution: $R = R_0 \exp\left[\sqrt{\frac{\Lambda}{3}}\, t\right]$

accelerated expansion:

$$\dot{R} \sim \exp\left[\sqrt{\frac{\Lambda}{3}} t\right]$$

$$\ddot{R} > 0$$

(for matter dominated stage

$R \sim t^{2/3}$; $\dot{R} \sim t^{-1/3}$, $\ddot{R} < 0$

Radiation dominated Universe

(take again $k=0$, $\lambda=0$)

try $R = t^{\alpha}$:

$$\frac{\alpha^2}{t^2} = \frac{8\pi G}{3}\rho \quad \Big] \quad \frac{(\dot{R})^2}{R^2} = \frac{8\pi G}{3}\rho$$

$$\frac{2\alpha(\alpha-1)}{t^2} + \frac{\alpha^2}{t^2} = -\frac{8\pi G}{3}\rho \quad \leftarrow \quad 2\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} = -8\pi G \cdot p \quad \left(\text{since } p = \frac{\rho}{3}\right)$$

equation for α: $2\alpha^2 + 2\alpha^2 - 2\alpha = 0$

or $2\alpha^2 - \alpha = 0$, $\alpha = 0$ and $\alpha = \frac{1}{2}$

So, the solution is:

$$R = R_0 \left(\frac{t}{t_0}\right)^{1/2}$$

$$\rho = \frac{3}{32\pi G t^2}$$

Summary of different behaviours

(i) Radiation dominated universe:

$$p = \frac{\rho}{3}, \quad \rho = \frac{3}{32\pi G}\frac{1}{t^2}, \quad R = R_0\left(t/t_0\right)^{1/2},$$

$$H = \frac{1}{2t}, \quad \rho \sim \frac{1}{R^4}$$

(ii) Matter dominated universe:

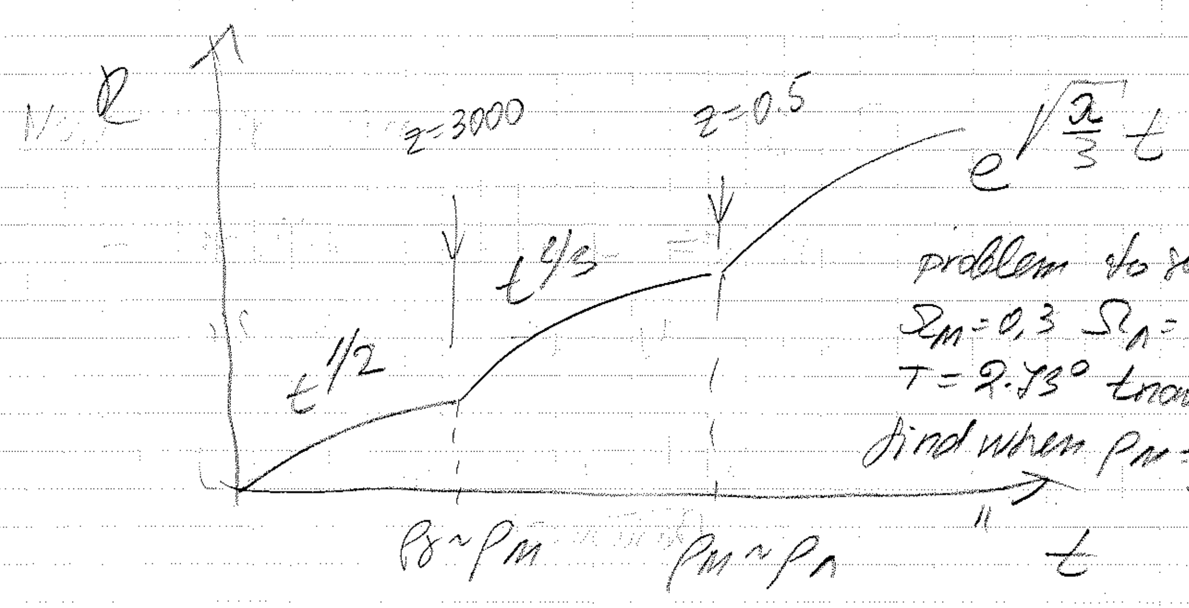
$$p = 0, \quad \rho = \frac{1}{6\pi G}\frac{1}{t^2}, \quad R = R_0\left(t/t_0\right)^{2/3}$$

$$H = \frac{2}{3t}, \quad \rho \sim \frac{1}{R^3}$$

(iii) Λ- dominated universe:

$$p = -\rho, \quad \rho = -p = \text{const}, \quad R = R_0 e^{Ht},$$

$$H = \sqrt{\frac{8\pi G\rho}{3}}, \quad \rho = \text{const.}$$



problem to solve:
$\Omega_M = 0.3$ $\Omega_\Lambda = 0.7$
$T = 2.73°$ $t_{now} = 1.4 \cdot 10^{10}$ years
find when $\rho_M = \rho_\Lambda$
" $\rho_\gamma = \rho_M$

Big Bang theory   Gamov

Common feature of radiation and matter dominated regimes: singularity, i.e. $\rho \to \infty$ when $t \to 0$

What happens there is unknown,

at $\rho \sim \frac{1}{G_N^2} \sim (10^{19}\ GeV)^4$ gravity is quantum

– open problem

However: if ρ is so large, then particles were very close to each other $\Rightarrow$ thermal equilibrium

Gamov's predictions:

- Cosmic background radiation:
universe was dense and hot and was cooling down - there were initially many photons,
Then plasma was neutralized at T ~ fraction of [13,6 eV] and the universe became transparent for radiation

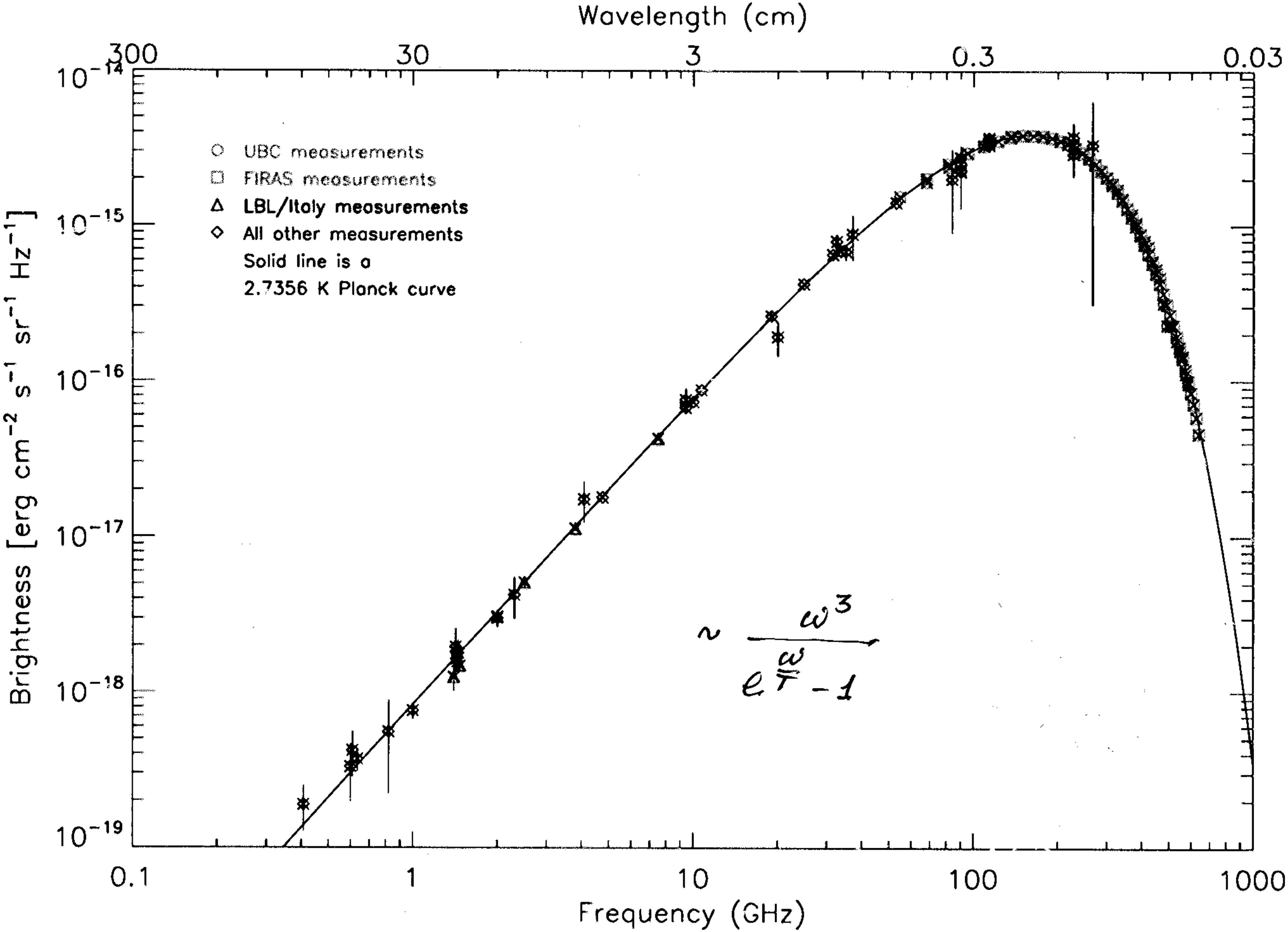
Prediction of the temperature:

$$\rho = \frac{\pi^2}{30} g T^4 \qquad (g=2 \text{ for photon})$$

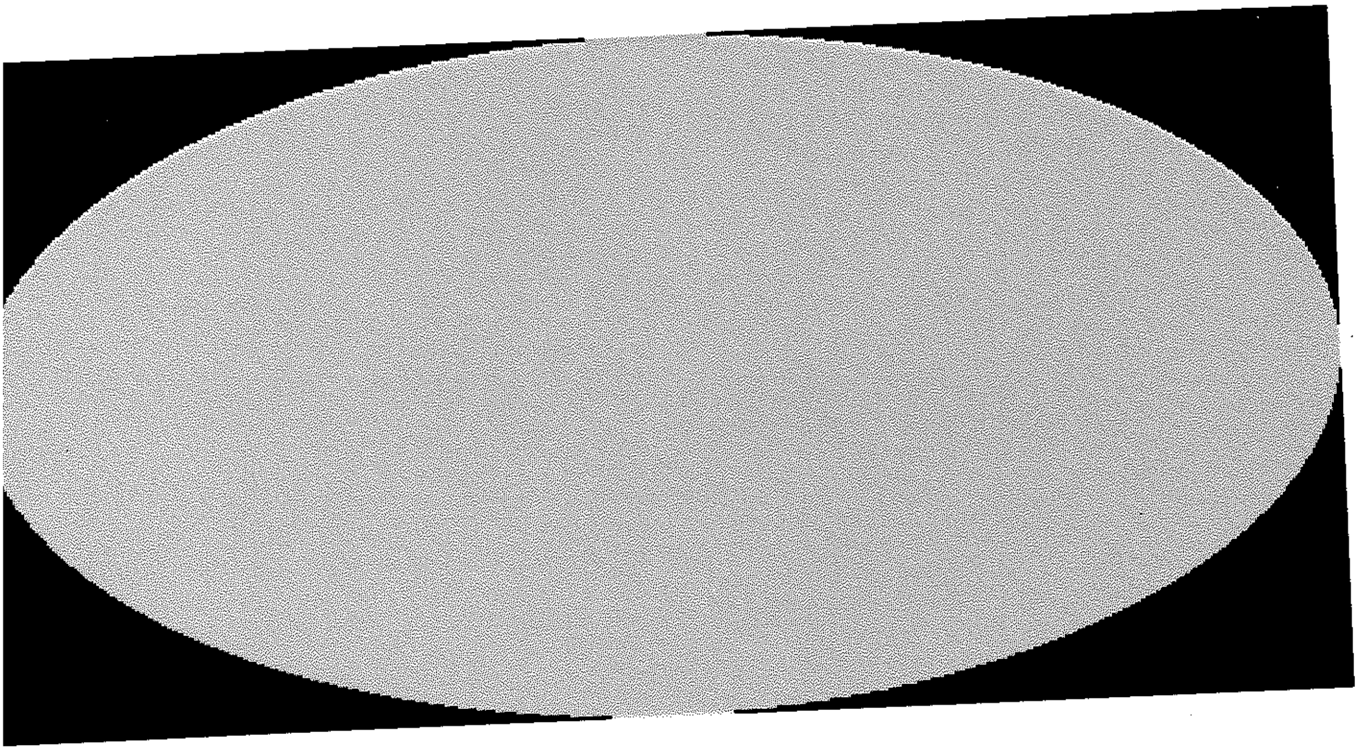
$$\text{so, } \frac{\pi^2}{30} g T^4 = \frac{3}{32\pi G t^2},$$ and to find

a temperature just insert there the present age of the universe

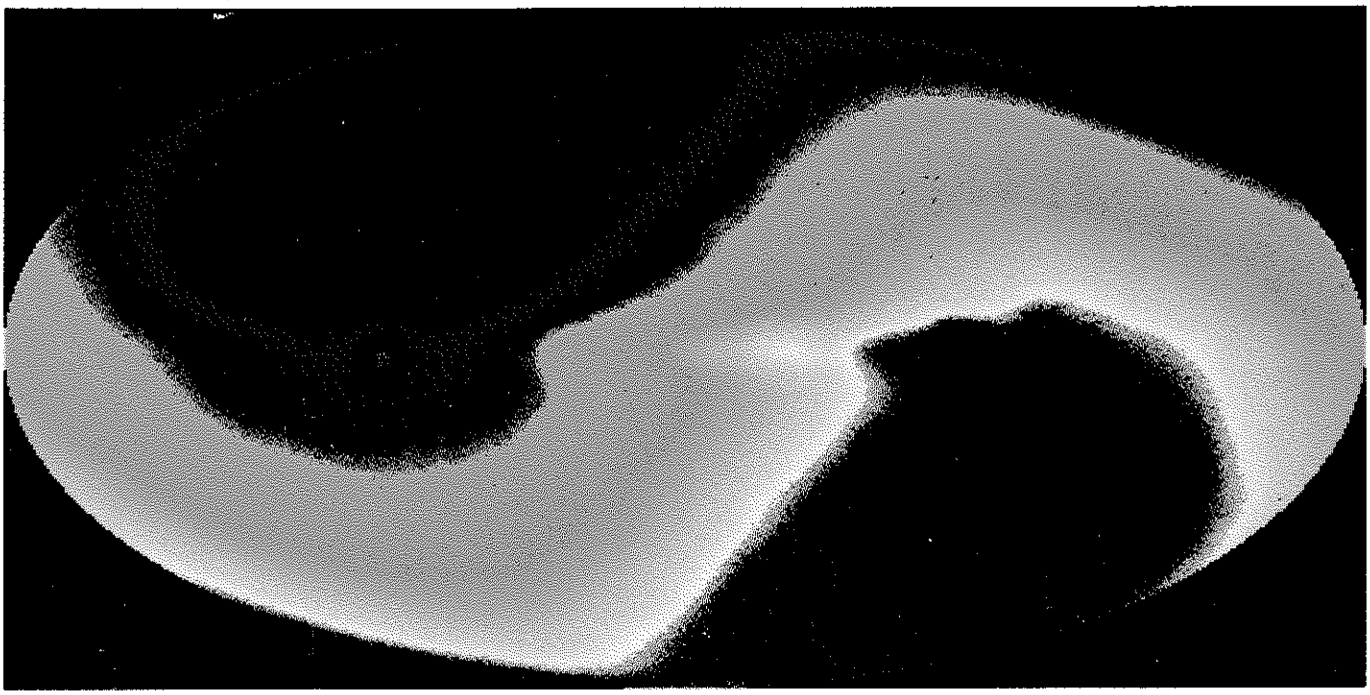
Was discovered by chance by Penzias and Wilson in 1965 (was seached by Peebles



$$\sim \frac{\omega^3}{e^{\frac{\omega}{T}} - 1}$$

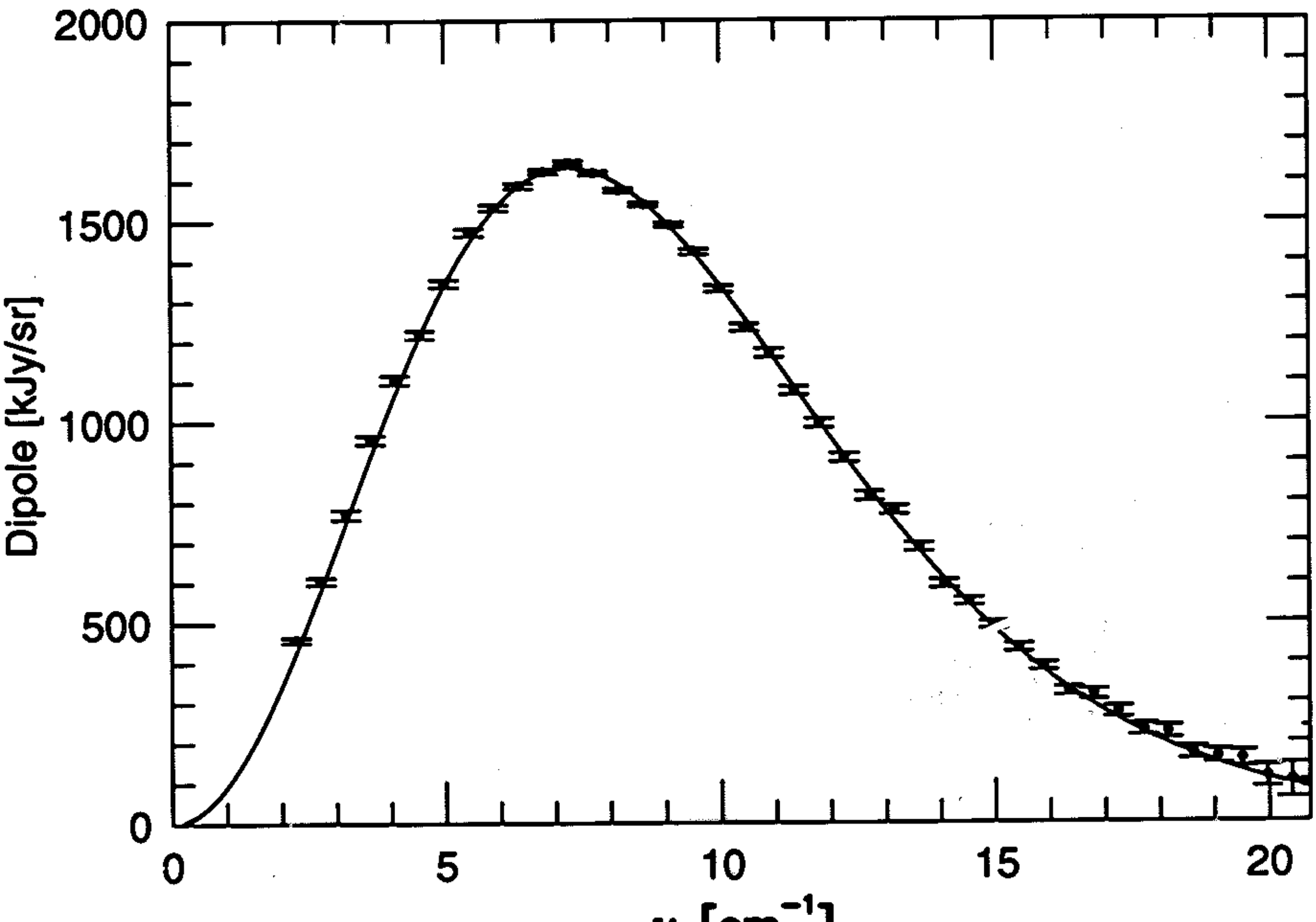


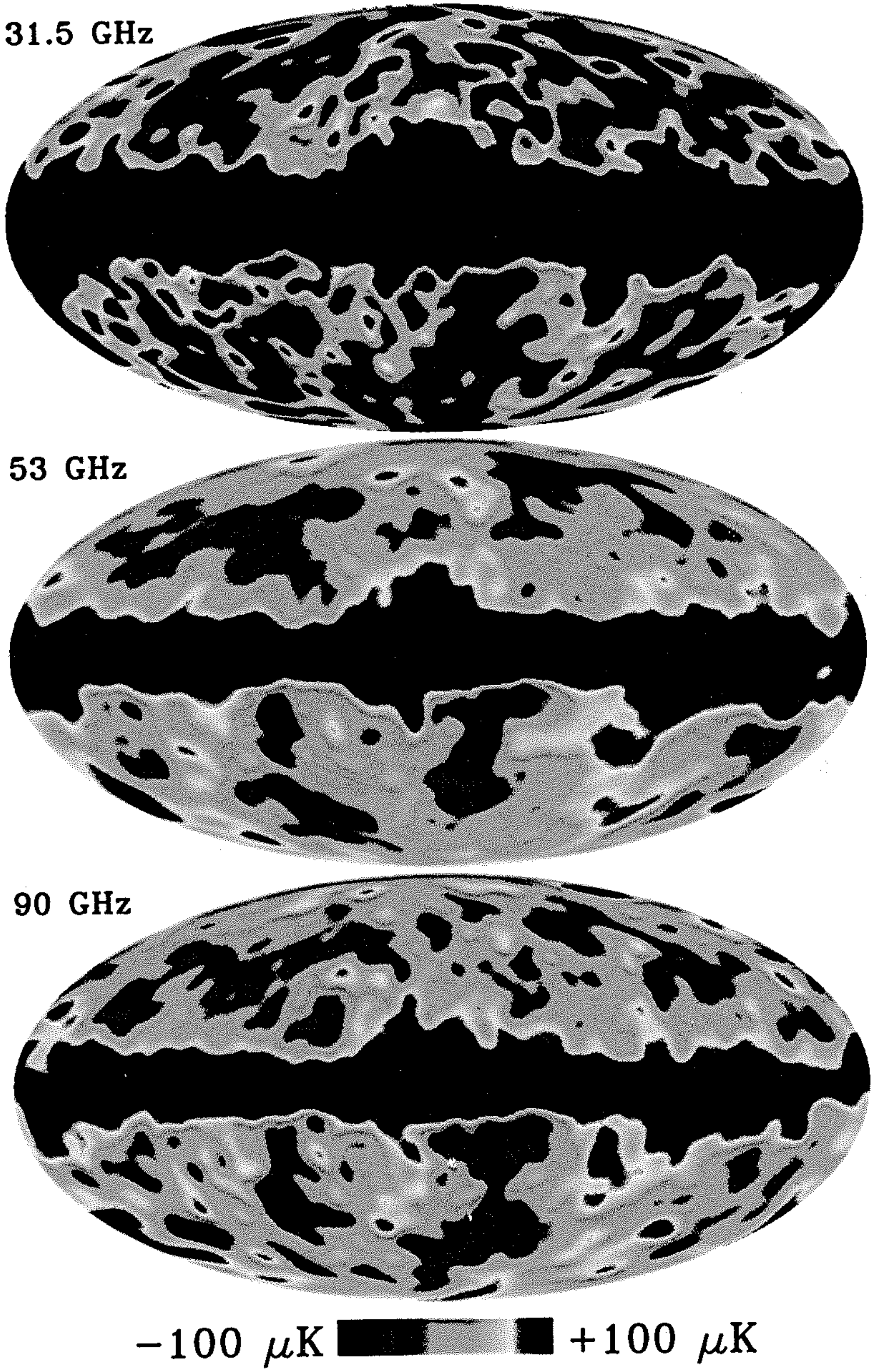
CMB, first approximation (isotropic)



CMB, 2nd approximation
(dipole), $\frac{\delta T}{T} \simeq 1.23 \cdot 10^{-3}$

$v = 371 \pm 0.5$ km/s


2000
1500
1000
500
0
Dipole [kJy/sr]
0
5
10
15
20
ν [cm⁻¹]


31.5 GHz
53 GHz
90 GHz
−100 μK
+100 μK

(12)

- Another prediction:
  abundances of light elements
  Li^7 D, He^3, He^4 – nucleosynthesis
  Alpher Bethe Gamow

will study later.

Thermal equilibrium -

very poweful statement

everything can be computed via very few parameters : Temperature and chemical potentials

Quantum physics:

density matrix

$$\hat{\rho} = \frac{1}{Z} \exp\left(-\frac{\hat{H}}{T} + \mu_i \hat{N}_i\right) \quad ; \quad [\hat{N}_i, \hat{H}] = 0$$

Z - statistical sum, $Tr\,\hat{\rho} = 1$

T - temperature ; μ - chemical potentials

N_i: e.g. baryon number, lepton number

$$\frac{\hbar}{i}\frac{d}{dt} N_i = [\hat{H}, \hat{N}_i] = 0 \Rightarrow \hat{N}_i \text{ - conserved numbers.}$$

Classical physics

$$\rho = \frac{1}{Z} \exp\left(-\frac{H(p,q)}{T} + \mu_i N(p,q)\right)$$

$$\{H, N\} = 0$$

1st approximation to equilibrium physics - free (non-interacting particles) number densities:

$$n = \frac{g}{(2\pi)^3} \int d^3\vec{p}\; f(\vec{p})$$

energy densities:

$$\rho = \frac{g}{(2\pi)^3} \int E(\vec{p})\, f(\vec{p})\, d^3p$$

$$f(p) = \frac{1}{e^{\frac{E-\mu}{T}} \pm 1} \qquad \begin{array}{l} + : \text{Fermi} \\ - : \text{Bose} \end{array}$$

relativistic limit : $T \gg m$, $T \gg \mu$:

$$\rho = \begin{cases} \frac{\pi^2}{30}\, g T^4 & \text{Bose} \\ \frac{7}{8}\frac{\pi^2}{30}\, g T^4 & \text{Fermi} \end{cases}$$

$$n = \begin{cases} \frac{\zeta(3)}{\pi^2}\, g T^3 & \text{Bose} \\ \frac{3}{4}\frac{\zeta(3)}{\pi^2}\, g T^3 & \text{Fermi} \end{cases}$$

$$p = \frac{\epsilon}{3}$$

total energy density:

$$\rho = \frac{\pi^2}{30} g_* T^4$$

$$g_* = \sum_{\text{bosons}} + \frac{7}{8} \sum_{\text{fermions}}$$

$$H = 1.66\, g_*^{1/2} \frac{T^2}{M_{pl}} \equiv \frac{T^2}{M_0}\,; \quad M_0 = \frac{M_{pl}}{1.66\, g_*^{1/2}}$$

$$t = 0.301\, g_*^{-1/2} \frac{M_{pl}}{T^2} = \frac{1}{\left[T\,[\mathrm{MeV}]\right]^2}\ \text{sek}.$$

Non-relativistic case:

$T \ll m \Rightarrow$

number density

$$n \approx g \left(\frac{mT}{2\pi}\right)^{3/2} \exp\left(-\frac{m-\mu}{T}\right)$$

energy density

$$\rho = m n$$

pressure

$$P = n T \ll \rho$$

Also, entropy density, relativistic case

$$s = -\int \frac{d^3k}{(2\pi)^3}\, n \log n =$$

$$= \frac{2\pi^2}{45} T^3 \begin{cases} 1 & \text{Bose} \\ 7/8 & \text{Fermi} \end{cases}$$