Relativity and Cosmology II

Lecture #3

Main points of #2

- Found Friedmann equations:

\[ \frac{\dot{R}^2}{R^2} + \frac{k}{R^2} - \frac{\dot{R}}{3} = \frac{8\pi G}{3} \rho \]

\[ 2 \frac{\ddot{R}}{R} + \frac{\dot{R}}{R} - \frac{k}{R^2} - \lambda = -8\pi G \rho \]

To find evolution of the universe, they have to be supplemented by equation of state, i.e.,

\[ \rho = \frac{p}{\gamma} \text{ (relativistic matter)} \]

\[ \rho = 0 \text{ (non-relativistic matter)} \]

- Derived F. equations for non-relativistic case, \( \lambda = k = \rho = 0 \), from Newton dynamics.

- Considered Einstein static universe, it is required, \( \lambda = 8\pi G \rho \).

- Considered Friedmann solution,

\[ p = 0, \ k = 0, \ A = 0 \]

\[ R = R_0 \left(1 + \frac{t^2}{R_0^2}ight)^{1/2} \]

\[ \rho = \frac{1}{6\pi G t^2} \]
found that the distances in expanding universe (flat case) change as:

\[ l \sim R(t) \Rightarrow \dot{l} \sim \dot{R}(t) \Rightarrow \frac{\dot{l}}{l} = \frac{\dot{R}}{R} \Rightarrow \]

\[ \frac{\dot{l}}{l} = \dot{H} ; \quad H = \frac{\dot{R}}{R} = 2 \frac{\dot{R}}{3R} \quad \text{for } \rho = 0 \]

\[ \text{prediction - solution} \]

\[ \text{red shift :} \]

\[ w' = (1 - \nu)w \quad [\text{Doppler effect}] \]

can be observed, if we have "standard candles" - certain type of similar stars

Today:

- red shift, FRW metric
- luminosity distance, FRW metric
Red shift, FRW metric

Let coordinates of the star are:

\[ \theta = \frac{\pi}{2}, \quad \phi = 0, \quad \bar{r} = \bar{r}_0 \]

They are all fixed.

Receive light in \( \bar{r} = 0 \) at \( t_0 \).

Light is emitted in \( \bar{r} = \bar{r}_0 \) at \( t_0 \).

Visualization, 2D sphere

We are here

\[ t = t_0 \]

"Size of universe"

\[ R(t_0) \]
equation for light propagation:
\[ ds^2 = 0 \Rightarrow dt^2 - R^2(t) \frac{dr^2}{1 - kr^2} = 0 \]

\[
\int_{t_1}^{t_2} \frac{dt}{R(t)} = \int_0^{r_2} \frac{dr}{\sqrt{1 - kr^2}} \begin{cases} \arcsin r_1 & k = 1 \\ \arcsinh r_1 & k = -1 \\ \text{the same} & k = 0 \end{cases}
\]

\[ \equiv f(r_1) \]

**Pulses:**
\[ t_0, t_0 + \delta t_0 \Rightarrow \]
\[ t_1, t_1 + \delta t_1 \]

\[ \int_{t_1 + \delta t_1}^{t_0 + \delta t_0} \frac{dt}{R(t)} = f(r_1) - \text{the same} \Rightarrow \]

\[ \frac{\delta t_0}{R(t_0)} = \frac{\delta t_1}{R(t_1)} \quad \text{frequency } \omega \sim \frac{1}{\delta t} \Rightarrow \]
\[
\frac{1}{w_2 R(t_2)} = \frac{1}{w_1 R(t_1)} \Rightarrow
\]

\[
\frac{w_0}{w_1} = \frac{R(t_1)}{R(t_2)} \quad w_0 = w_1 \frac{R(t_1)}{R(t_2)}
\]

what we measure = what was emitted

Red shift, definition:

\[
z = \frac{\lambda_0 - \lambda_1}{\lambda_0} = \frac{R(t_0)}{R(t_1)} - 1
\]

observed < emitted

Physical picture:
red shift:
\(z > 0\) if the universe is expanding
\(z < 0\) if the universe is collapsing (blue shift)

Measurements of distances:

(i) comparison between absolute luminosity and apparent luminosity

(ii) if exact size of object is known, we can find the distance
(ii) find parallax - change of position of the object due to change of its visible position due to Earth motion

Consider in more detail 1st method (used for cosmological distances)

\[ r = r_1, \text{ source} \]

\[ \text{telescope, radius } b, \text{ area } S = \pi b^2 \]

"geometrical" fraction of energy, emitted at \( t = t_1 \) and passing through telescope:

space part of the interval at \( t = t_0 \)

\[ dl^2 = R^2(t_0) \left[ \frac{d\nu^2}{1 - \nu^2} + \frac{\nu^2}{c^2} d\Omega^2 \right] \]

total area of light front:

\[ S_{tot} = \int R^2(t_0) \frac{\nu^2}{c^2} d\Omega^2 = 4\pi r_1^2 R^2(t_0) \]

fraction: \( S / S_{tot} \)
Energy flux registered at telescope:

\[ P = \Delta \cdot \frac{S}{4\pi R^2(t_0) \cdot \bar{\nu}^2} \cdot \frac{\hbar \omega_0}{\hbar \omega_1} \cdot \frac{8t_1}{8t_0} \]

Total energy emitted

\[ \frac{\hbar \omega_0}{\hbar \omega_1} = \frac{R(t_1)}{R(t_0)} \]

Energy of emitted photon

Ratio between pulses emitted \((8t_1)\) and absorbed \((8t_0)\)

\[ \frac{8t_1}{8t_0} = \frac{R(t_1)}{R(t_0)} \]

giving

\[ P = \Delta \left( \frac{R^2(t_1)}{R^2(t_0)} \right) \frac{S}{4\pi R^2(t_0) \bar{\nu}^2} \]
apparent luminosity, power per unit of surface,

\[ L = \frac{\Delta R^2(t)}{4\pi R_0^2(t) R_1^2} \]

in flat space we would get:

\[ d = \frac{\Delta}{4\pi d^2} \quad d: \text{"photometric" distance} \]

\[ \Rightarrow \frac{\Delta}{4\pi d^2} = \frac{\Delta R^2(t)}{4\pi R_0^2(t) R_1^2} \Rightarrow d = R^2(t) \frac{R_1^2}{R(t)} \]

let us find now relation between red shift and \( d \):

Then in this equation we have to add:

\[ \frac{R(t_0)}{R(t)} - 1 = z \]

and \( \int_{t_1}^{t_0} \frac{dt}{R(t)} = f(z) \), defined at page 4.
This completes the system of equations determining $z$ as a function of $d$.

How does it work, linear approximation:

Let $r_1$ be small, and thus $t_1$ is close to $t_0$.

Then $f(r) \approx r$; and

$$\frac{t_0 - t_1}{R} \approx r_1$$

$$z = \frac{R(t_0)}{R(t_1)} - 1 = \frac{R(t_0)}{R(t_0) + \frac{R(t_0)}{(t_1 - t_0)}} - 1$$

$$\approx + \frac{\dot{R}(t_0) (t_0 - t_1)}{R(t_0)} \approx \frac{\dot{R}(t_0)}{R(t_0)} \cdot \frac{r_2 R(t_0)}{R(t_0)}$$

and $d \propto R(t_0) \cdot r_2 \Rightarrow$

$$z \approx \frac{\dot{R}(t_0)}{R(t_0)} \cdot d = H \cdot d$$

Red shift is proportional to

distance

**Hubble law**

In terms of velocity:

$$v = \frac{R}{R} \cdot d$$

If Doppler effect is used.
Relativity and cosmology II

Lecture #4

Main points of #3

- Relations between luminosity distance and red shift

\[ d = \frac{R^2(t_0)}{R(t_i)} \quad \frac{R(t_0)}{R(t_i)} \quad (\ast) \]

\[ z = \frac{R(t_0)}{R(t_i)} - 1 \quad (\ast\ast) \]

\[ \int_{t_0}^{t} \frac{dt}{R(t)} = \int_0^\infty \text{arctan} \frac{1}{\sqrt{1 - \kappa}} \quad \kappa = \pm 1 \quad (\ast\ast\ast) \]

\[ \int_0^\infty \text{arcosh} \frac{1}{\sqrt{1 - \kappa}} \quad \kappa = 0 \quad (\ast\ast\ast) \]

- Hubble law:

\[ z = H_0 t \]

Plan for today

- Luminosity distance - red shift relation for general case

- Critical density

- Content of the universe, abundances

- Cosmological parameters from supernova

- Properties of universe as a function of \( \Omega_m, \Omega_L \)
To find \( d-z \) relation for general case, find \( \bar{v}_1 \) from (**):

\[
\bar{v}_1 = \left\{ \begin{array}{l}
\int_{t_1}^{t_0} \frac{dt}{R(t)} \\
\sin \left( \int_{t_1}^{t_0} \frac{dt}{R(t)} \right) \end{array} \right\} = \bar{v}_2 \left( t_2, t_0 \right)
\]

depending on \( k = +1, 0 \) and \(-1\)

Then

\[
d(z) = R(t_0)(1+z) \bar{v}_2 \left( t_2, t_0 \right),
\]

and we have to find \( t_2 \) from (**)

We will solve this problem today, and find how \( t_2 \), etc depend on the content of the universe.

- First: introduce "vacuum" energy and pressure
- Second: introduce critical density
- Third: introduce abundances of different components of the universe
Critical density

Let us rewrite F. equations in somewhat different notations:

"vacuum energy density":

\[ \rho_v = \frac{\Lambda}{8\pi G} \]

"total energy density"

\[ \rho_{\text{tot}} = \rho + \rho_v \]

"total pressure"

\[ P_{\text{tot}} = P + P_v \quad \text{and} \quad P_{\text{v}} = -\rho_v \]

and replace \( R \) by \( H R \), where \( H \) is the Hubble constant.

Friedman equations:

\[
\begin{cases}
H^2 + \frac{\mathcal{K}}{R^2} = \frac{8\pi G}{3} \rho_{\text{tot}} \\
\frac{d\rho}{dt} = -3H(\rho + P) = \text{energy conservation}
\end{cases}
\]

Definition: critical density

\[ \rho_c = \frac{3H_0^2}{8\pi G} \]
\( p \) depends on time and can be found from the knowledge of the Hubble constant.

Value: \( p_c = 1.88 \times 10^{-39} \text{ g/cm}^3 \)

With \( H_0 = 100 \text{ km/s/Mpc} \)

\( p \) can be found by other means:

\[
\begin{align*}
\text{if } p > p_c & \Rightarrow k > 0 \text{, universe is closed} \\
\text{if } p = p_c & \Rightarrow k = 0 \text{, universe is flat} \\
\text{if } p < p_c & \Rightarrow k < 0 \text{, universe is open}
\end{align*}
\]

The relation between \( p \) and \( p_c \) is also important for the evolution of the universe.

Example: Consider a matter-dominated universe with \( p = 0 \) and \( k = 0 \).

Equation for \( R \), valid also for \( k \neq 0 \):

\[
\frac{\ddot{R}}{R} = -\frac{4\pi G}{3} p \Rightarrow \frac{\ddot{R}}{R} = -\frac{4\pi G}{3} \rho R
\]

Multiply by \( \dot{R} \) and use the fact that

\[
pR^3 = \text{const} = p_0 R_0^3, \quad p = \frac{p_0 R_0^3}{R^3}
\]
\[
\frac{d}{dt}\left(\frac{1}{2} R^2\right) = \ddot{R} R = -\frac{4\pi G}{3} \frac{P_o R_o^3}{R^3} \cdot R \cdot \ddot{R} \Rightarrow
\]

\[
\frac{1}{2} \dot{R}^2 - \frac{4\pi G}{3} \frac{P_o R_o^3}{R} = \text{Const} = -\frac{4\pi G}{3} R_o^3 (P_o - P_c)
\]

**Mechanical Interpretation:**

Particle of mass \(m\) moves in potential \(U(R)\) with total energy \(E_{\text{tot}}\):

\[
U(R) = -\frac{4\pi G}{3} \frac{P_o R_o^3}{R}
\]

\[
E_{\text{tot}} = -\frac{4\pi G}{3} R_o^3 (P_o - P_c)
\]

- For \(E_{\text{tot}} \geq 0\):
  - Possible behaviours:
    1. \(P_o < P_c \Rightarrow E_{\text{tot}} > 0 \Rightarrow R(t) \to \infty \text{ as } t \to \infty\) \{expands forever\}
    2. \(P_o = P_c \Rightarrow E_{\text{tot}} = 0 \Rightarrow R(t) \to \infty \text{ as } t \to \infty\)
    3. \(P_o < P_c \Rightarrow E_{\text{tot}} < 0 \Rightarrow R_{\text{max}} = \frac{2\pi G P_o}{P_o - P_c}\) bounces back and goes to infinity
General content of the Universe

Let universe contain

- radiation with \( p_r = \frac{p_\gamma}{3} \)
- non-relativistic matter
  (+ dark matter, to be discussed later), \( p_m = 0 \)
- cosmological constant

\[ p_\Lambda = -p_\Lambda \]

How did the universe evolve?

Definition: abundances

\[ \Omega_\gamma = \frac{p_r}{p_{\text{ent}}}, \quad \Omega_m = \frac{p_m}{p_{\text{ent}}}, \quad \Omega_\Lambda = \frac{p_\Lambda}{p_{\text{ent}}} \]

from \( H^2 = \frac{k}{R^2} = \frac{8\pi G}{3} p_{\text{tot}} \quad \Rightarrow \)

\[ \frac{3H^2}{8\pi G} + \frac{3}{8\pi G} \frac{k}{R^2} = p_{\text{tot}} \quad \text{or} \]

\[ p_0 \]

\[ 1 = \Omega_\gamma + \Omega_m + \Omega_\Lambda + \Omega_k \]

where \( \Omega_k = -\frac{k}{p_0 H_0^2} \), curvature contribution

Behaviour of \( p_{\text{tot}} \):
\[ p_{\text{tot}} = \rho c \left[ \Omega_{\Lambda} + \Omega_{M} \left( \frac{R(H_0)}{R(t)} \right)^3 + \Omega_{\gamma} \left( \frac{R(H_0)}{R(t)} \right)^4 \right] \]

part, associated with cosmological constant does not change.

part, associated with matter:

\[ \frac{d}{dt} (\rho m R^3) = 0 \Rightarrow \rho m \propto \frac{1}{R^3(t)} \]

part, associated with radiation:

\[ \frac{d}{dt} (\rho \gamma R^3) + \rho \gamma \frac{d}{dt} \frac{R^3}{H} = 0 \]

since \( a = \frac{R}{t} \), \( \frac{d}{dt} (\rho \gamma R^3) = 0 \Rightarrow \)

\[ \rho \gamma \propto \frac{1}{R^4} \]

easy to understand:

- # of photons \( \propto \frac{1}{R^3} \)
- energy of photon, \( w = \frac{1}{R} \)
Convenient form of Friedmann equation:

\[
H^2 \equiv \left( \frac{R}{R} \right)^2 = \frac{8 \pi G}{3} \rho_{\text{tot}} - \frac{k}{R^2} = \frac{8 \pi G}{3} (\rho_{\text{tot}} + \rho_k)
\]

Last term, do have similar notations:

\[
\rho_k = -\frac{k}{R^2} \frac{3}{8 \pi G} \equiv \rho_c \Omega_k \frac{R(t_0)^2}{R(t)^2}
\]

"Curvature abundance"

\(\Omega_k\) is not independent,

Sum rule:

\[
1 = \Omega_n + \Omega_m + \Omega_r + \Omega_k
\]

to derive the sum rule, just divide first equation at this page by \(H^2\).

Final form of Friedmann equation we will work:

\[
\left( \frac{\dot{R}}{R} \right)^2 = \frac{8 \pi G}{3} \rho_c \left[ \Omega_n + \Omega_m \left( \frac{R(t_0)}{R} \right)^3 + \Omega_r \left( \frac{R(t_0)}{R(t)} \right)^4 \right] + \Omega_k \left( \frac{R(t_0)}{R(t)} \right)^2
\]

Notation: \(R(t_0) = R_0\)
This equation can be also written in a form, which has a simple mechanical analogy:

\[ \frac{1}{2} \dot{R}^2 + U(R) = 0, \quad \text{where} \]

\[ U(R) = -\frac{4\pi G}{3} \rho_c \left( \Omega_\Lambda \cdot R^2 + \Omega_m \frac{R_0^3}{R} ight) 
+ \Omega_\gamma \left( \frac{R_0^4}{R^2} + \Omega_k R_0^2 \right) \]

it describes a motion of particle with zero energy in potential \( U(R) \)

Let us integrate now \((x x x x)\)

notations: \( x = \frac{R(t)}{R(t_0)} \); \( R(t) = R_0 \cdot x \)

\[ \Omega_\Lambda + \Omega_m \frac{1}{x^3} + \Omega_k \frac{1}{x^2} + \Omega_\gamma \frac{1}{x^4} \equiv A^2(x) \]

Eq \((x x x x)\) for \( x \):

\[ \frac{x^2}{\dot{x}^2} = H_0^2 A^2(x) \Rightarrow \frac{dx}{dt} = H_0 A(x) \cdot \dot{x} \]

\[ dt = \frac{dx}{H_0 A(x) \cdot \dot{x}} \]; \( \frac{dt}{dx} = \frac{1}{H_0 A(x) \cdot \dot{x}} \)
This allows us to get immediately from expression on page 2 for d(t):

\[
\int_{t_1}^{t_0} dt \cdot \int_{t_1}^{t_0} \frac{dt}{dx} \frac{dx}{R_0 x} = \int_{t_1}^{t_0} dx \frac{1}{x^2 R_0 H_0 A(x)}
\]

\[
= \int_{1+z}^{1} \frac{dx}{x^2 R_0 H_0 A(x)}
\]

to corresponds to \( x = 1 \)

t_1 corresponds to \( x = \frac{1}{1+z} = \frac{R(t_1)}{R(t_0)} \)

For example, for \( z = 0 \) we get:

\[
d(t) = \frac{1+z}{H_0} \int_{1}^{1+z} \frac{dx}{x^2 A(x)}
\]

Equation, which can unify all cases:

\[
d(t) = \frac{1+z}{H_0 \Omega_k^{\frac{1}{2}}} \sinh \left[ \Omega_k^{\frac{1}{2}} \int_{1}^{1+z} \frac{dx}{x^2 A(x)} \right]
\]

Since, e.g.,

for \( k = +1 \) \( R_0 H_0 = \Omega_k \)

taking formally \( \Omega_k \to 0 \), we get flat case, taking \( \Omega_k < 0 \) we replace \( \sinh \to \sin \)
By product of computation:

ease of the Universe.

Take the formula at the bottom of page 3, and integrate it from $t = 0$ to $t_0$.

$t = 0$ corresponds to $a = 0$ (since $R = 0$)

$t = t_0$ corresponds to $a = 1$

$$t = \frac{1}{H_0} \int_{0}^{1} \frac{dx}{a A(a)}$$

How do we use $d(\tau)$ for determination of content of the Universe:

1) measure curve $d(\tau)$

2) find parameters $\Omega_x, \Omega_m$ and $\Omega_{\Lambda}$ from fitting this curve by expression for $d(\tau)$

In practice: $\Omega_x \ll \Omega_m, \Omega_{\Lambda} \ll \Omega_m$

and can be neglected.

$\Lambda \text{CDM model}$
standard candels: 

SN Ia, 

Instrument: Hubble space telescope

Supernova Cosmology Project 
Perlmutter et al. (1998)

\[(\Omega_M, \Omega_A) = (0, 1), (0.5, 0.5), (1, 0), (1.5, 0.5), (2, 0)\]

Calan/Tololo 
(Hamuy et al., 
A.J. 1996)

In flat universe: \(\Omega_M = 0.28 \pm 0.085 \text{ statistical} \pm 0.05 \text{ systemat}^2\)

Prob. of fit to \(\Lambda = 0\) universe: 1%

astro-ph/9812133
Supernova Cosmology Project

Knop et al. (2003)
Spergel et al. (2003)
Allen et al. (2002)

$\Omega_\Lambda$

$\Omega_M$

No Big Bang

Supernovae

CMB

Clusters

expands forever

recollapses eventually

closed

flat

open
Supernova Cosmology Project
Knop et al. (2003)

\( \Omega_\Lambda \)

\( \Omega_M \)

No Big Bang

68%, 90%, 95%, 99%

Accelerating
Decelerating

Expands Forever
Recollapses Eventually

Open
Flat
Closed
- Flat: $\Omega_k = 0$
- Closed: $\Omega_k < 0$
- Open: $\Omega_k > 0$

- Recollapses eventually: $R(t) \rightarrow 0$ for some $t > t_0$ ($\dot{R}(t) = 0$ for some $t > t_0$)

- Expands forever: $R(t) > 0$

- Decelerating: $\ddot{R} < 0$
- Accelerating: $\ddot{R} > 0$

- No Big Bang: $R > 0$ for all $t < t_0$ (no singularity)