Relativity and Cosmology II

Lecture #3

Main points of #2

- Found Friedmann equations:

$$\frac{\dot{R}^2}{R^2} + \frac{k}{R^2} - \frac{\Lambda}{3} = \frac{8\pi G}{3}\rho$$

$$2\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + \frac{k}{R^2} - \Lambda = -8\pi G \cdot p$$

To find evolution of the universe, they have to be supplemented by equation of state, e.g.

$$p = \frac{\rho}{3} \quad \text{(relativistic matter)}$$

$$p = 0 \quad \text{(non-relativistic matter)}$$

- Derived F. equations for non-relativistic case, $\Lambda = k = p = 0$, from Newton dynamics

- considered Einstein static universe
it is required, $\Lambda = 8\pi G \rho$

- considered Fridman solution,

$$p = 0, \; k = 0, \; \Lambda = 0$$

$$R = R_0 \, t^{2/3}$$

$$\rho = \frac{1}{6\pi G} \frac{1}{t^2}$$

found that the distances in expanding universe (flat case) change as:

$$\ell \sim R(t) \Rightarrow$$

$$\dot{\ell} \sim \dot{R}(t) \Rightarrow \quad \frac{\dot{\ell}}{\ell} = \frac{\dot{R}}{R} \Rightarrow$$

$$\dot{\ell} = \ell H; \quad H = \frac{\dot{R}}{R} = \frac{2}{3t}, \text{ for } p=0$$

solution

prediction -
red shift :

$$\omega' = (1-v)\omega \quad [\text{Doppler effect}]$$

can be observed, if we have „standard" candels -
certain type of similar stars

Today:

- red shift, FRW metric
- luminosity distance, FRW metric

Red shift, FRW metric

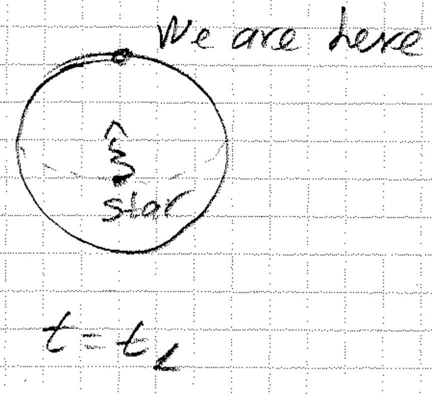
Let coordinates of the star are:

$$\theta = \frac{\pi}{2}, \quad \varphi = 0, \quad \bar{r} = \bar{r}_1$$

They are all fixed.

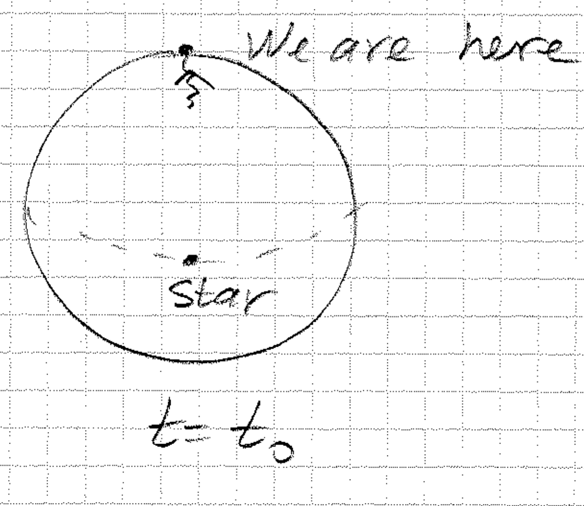
receive light in $\bar{r} = 0$ at t_0
light is emitted in $\bar{r} = \bar{r}_1$ at t_1

visualuzation, 2d sphere



$t = t_1$

Size of Universe
$R(t_1)$



$t = t_0$

Size of Universe
$R(t_0)$

equation for light propagation:

$$ds^2 = 0 \Rightarrow dt^2 - R^2(t)\frac{d\bar{r}^2}{1-k\bar{r}^2} = 0$$

$$\int_{t_1}^{t_0}\frac{dt}{R(t)} = \int_0^{r_1}\frac{d\bar{r}}{\sqrt{1-k\bar{r}^2}} = \begin{cases}\arcsin r_1 & k=+1\\ r_1 & k=0\\ \operatorname{arcsh} r_1 & k=-1\end{cases}$$

$$\equiv f(r_1)$$

pulses: $t_0 \quad t_0+\delta t_0$ $\Rightarrow$

$t_1 \quad t_1+\delta t_1$

$$\int_{t_1+\delta t_1}^{t_0+\delta t_0}\frac{dt}{R(t)} = f(r_1) \text{ – the same } \Rightarrow$$

$$\frac{\delta t_0}{R(t_0)} = \frac{\delta t_1}{R(t_1)} \quad \text{frequency } \omega \sim \frac{1}{\delta t} \Rightarrow$$

$$\frac{1}{\omega_0 R(t_0)} = \frac{1}{\omega_1 R(t_1)} \Rightarrow$$

$$\frac{\omega_0}{\omega_1} = \frac{R(t_1)}{R(t_0)}; \quad \omega_0 = \omega_1 \frac{R(t_1)}{R(t_0)}$$

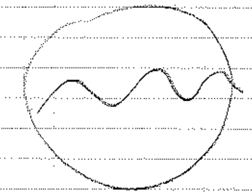
what we measure (ω_0) — what was emitted (ω_1)

Red shift, definition:

$$z = \frac{\lambda_0 - \lambda_1}{\lambda_1} = \frac{R(t_0)}{R(t_1)} - 1$$

observed (λ_0) — emitted (λ_1)

Physical picture:



red shift:
$z > 0$ if the universe is expanding

put harmonics on a sphere

$z < 0$ if the universe is collapsing (blue shift)

<u>Measuriments of distances</u>

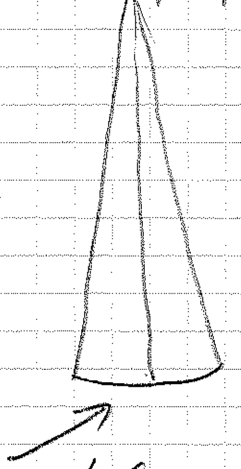
(i) comparison between „absolute luminocity" and apparent luminosity

(ii) if exact size of object is known, we can find the distance

(iii) find parallax – change of position of the object due to change of its ~~position~~ visible due to Earth motion

Consider in more detail 1st method (used for cosmological distances)

$\bar{r} = \bar{r}_1$, source



telescope, radius b, area $S = \pi b^2$

"geometrical" fraction of energy, emitted at $t = t_1$ and passing through telescope:

space part of the interval at $t = t_0$

$$dl^2 = R^2(t_0)\left[\frac{d\bar{r}^2}{1-k\bar{r}^2} + \bar{r}^2 d\Omega^2\right]$$

total area of light front:

$$S_{tot} = \int R^2(t_0)\,\bar{r}_1^2\, d\Omega^2 = 4\pi \bar{r}_1^2 R^2(t_0)$$

fraction: S / S_{tot}

Energy flux per unit time registered at telescope:

$$P = \underbrace{L}_{\text{total Energy emitted}} \cdot \frac{S}{4\pi R^2(t_0)\cdot \bar{r}_2^{\,2}} \cdot \underbrace{\frac{\overbrace{\hbar\omega_0}^{\text{energy of registered photon}}}{\hbar\omega_1}}_{\text{energy of emitted photon}} \cdot \underbrace{\frac{\delta t_1}{\delta t_0}}_{\text{ratio between pulses emitted } (\delta t_1) \text{ and absorbed } (\delta t_0)}$$

$$\frac{\hbar\omega_0}{\hbar\omega_1} = \frac{R(t_1)}{R(t_0)}$$

$$\frac{\delta t_1}{\delta t_0} = \frac{R(t_1)}{R(t_0)}$$

giving

$$P = L\left(\frac{R^2(t_1)}{R^2(t_0)}\right)\frac{S}{4\pi R^2(t_0)\,\bar{r}_1^{\,2}}$$

apparent luminosity: Power per unit of surface,

$$\ell = \frac{L R^2(t_1)}{4\pi R^4(t_0)\,\bar{r}_1^2}$$

in flat space we would get:

$$\ell = \frac{L}{4\pi d^2} \quad d\text{: „fotometric'' distance}$$

$$\Rightarrow \frac{L}{4\pi d^2} = \frac{L R^2(t_1)}{4\pi R^4(t_0)\,\bar{r}_1^2} \Rightarrow$$

$$\boxed{d = R^2(t_0)\,\frac{\bar{r}_1}{R(t_1)}}$$

let us find now relation between red shift and d:

Then to this equation we have to add:

$$\frac{R(t_0)}{R(t_1)} - 1 = z$$

and $\displaystyle\int_{t_1}^{t_0} \frac{dt}{R(t)} = f(\bar{r}_1)$, defined at page 4

This completes the system of equations determining z as a function of d

How does it work, linear approximation:

let r_1 is small, and thus t_1 is close to t_0

then $f(r) \approx r$; and

$$\frac{t_0 - t_1}{R} \approx r_1 ;$$

$$z = \frac{R(t_0)}{R(t_1)} - 1 = \frac{R(t_0)}{R(t_0) + \dot{R}(t_0)(t_1 - t_0)} - 1$$

$$\approx + \frac{\dot{R}(t_0)(t_0 - t_1)}{R(t_0)} \approx \frac{\dot{R}(t_0)}{R(t_0)} \cdot r_1 R(t_0)$$

but $d \approx R(t_0) \cdot r_1 \Rightarrow$

$$\boxed{z \approx \frac{\dot{R}(t_0)}{R(t_0)} \cdot d} = H \cdot d$$

red shift is proportional to distance

Hubble law

In terms of velocity:

$$v = \frac{\dot{R}}{R} \cdot d$$

if Doppler effect is used

Relativity and cosmology II

Lecture # 4

Main points of # 3

- relations between luminocity distance, and red shift

$$d = R^2(t_0)\,\frac{\bar{r}_1}{R(t_1)} = R(t_0)(1+z)\cdot \bar{r}_1 \qquad (*)$$

$$z = \frac{R(t_0)}{R(t_1)} - 1 \qquad (**)$$

$$\int_{t_1}^{t_0} \frac{dt}{R(t)} = \begin{cases} \arcsin \bar{r}_1 & k=+1 \\ \bar{r}_1 & k=0 \\ \operatorname{arcsh} \bar{r}_1 & k=-1 \end{cases} \qquad (***)$$

- Hubble law:

$$z = Hd; \qquad H = \frac{\dot{R}}{R}$$

Plan for today

- luminocity distance - red shift relation for general case
- critical density
- content of the universe, abundances
- cosmological parameters from supernova
- properties of universe as a function of Ω_M, Ω_Λ

To find d - z relation for general case, find $\bar{r}_1$ from (***):

$$\bar{r}_1 = \begin{cases} \sin\left(\int_{t_1}^{t_0} \frac{dt}{R(t)}\right) \\ \int_{t_1}^{t_0} \frac{dt}{R(t)} \\ \mathrm{Sh}\left[\int_{t_1}^{t_0} \frac{dt}{R(t)}\right] \end{cases} \equiv \bar{r}_1(t_1, t_0),$$

depending on $k = +1, 0$ and -1

Then

$$d(z) = R(t_0)(1+z)\,\bar{r}_1(t_1, t_0),$$

and we have to find t_1 from (**)

We will solve this problem today, and find how t_1, etc depend on the content of the Universe.

- first: introduce "vacuum" energy and pressure
- second - introduce critical density
- third - introduce abundances of different components of the Universe

Critical density

Let us rewrite F. equations in somewhat different notations:

„vacuum energy density":

$$\rho_\Lambda = \frac{\Lambda}{8\pi G}$$

„total energy density"

$$\rho_{tot} = \rho + \rho_\Lambda$$

„total pressure"

$$p_{tot} = p + p_\Lambda \ ; \quad p_\Lambda = -\rho_\Lambda$$

vacuum pressure

and replace $\dot{R}$ by HR, where H is the Hubble constant

Friedman equations:

$$\begin{cases} H^2 + \frac{K}{R^2} = \frac{8\pi G}{3}\rho_{tot} \\ \frac{d\rho}{dt} = -3H(\rho + p) \quad \Leftarrow \text{ energy conservation} \end{cases}$$

Definition: critical density $\rho_c \equiv \dfrac{3H_0^2}{8\pi G}$

(4)

ρ_c depends on time and can be found from the knowledge of the Hubble constant value : $\rho_c \approx 1.88 \cdot h^2 \cdot 10^{-29}\, g/cm^3$

with $H_0 = 100 h\ km/s \cdot Mpc$

ρ can be found by other means $\Rightarrow$

if $\begin{cases} \rho^{tot} > \rho_c \Rightarrow k > 0 & \text{universe is closed} \\ \rho^{tot} = \rho_c \Rightarrow k = 0 : & \text{universe is flat} \\ \rho^{tot} < \rho_c \Rightarrow k < 0 : & \text{universe is open} \end{cases}$

The relation between ρ and ρ_c also important for evolution of the universe.

Example : consider matter-dominated universe with $p = 0$ and $\Lambda = 0$

Equation for R, valid also for $k \neq 0$:

$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}\rho \Rightarrow \ddot{R} = -\frac{4\pi G}{3}\rho R$$

multiply by $\dot{R}$ and use the fact that

$$\rho R^3 = \text{const} = \rho_0 R_0^3,\ \rho = \frac{\rho_0 R_0^3}{R^3}$$

$$\frac{d}{dt}\left(\frac{1}{2}\dot{R}^2\right) = \dot{R}\ddot{R} = -\frac{4\pi G}{3} \cdot \frac{\rho_0 R_0^3}{R^3} \cdot R \cdot \dot{R} \Rightarrow$$

$$\frac{1}{2}\dot{R}^2 - \frac{4\pi G}{3}\frac{\rho_0 R_0^3}{R} = \text{Const} =$$

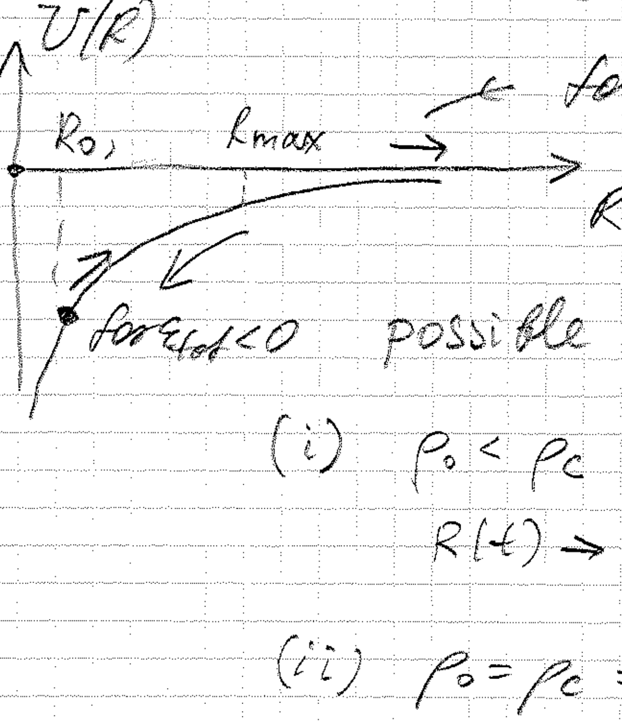
$$-\frac{4\pi G}{3} R_0^3 (\rho_0 - \rho_c)$$

mechanical interpretation:

particle of mass $= 1$ moves in potential

$U(R) = -\frac{4\pi G}{3}\frac{\rho_0 R_0^3}{R}$ with total energy

$$\mathcal{E}_{tot} = -\frac{4\pi G}{3} R_0^3 (\rho_0 - \rho_c)$$

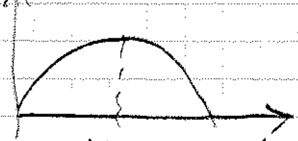


possible behaviours:

(i) $\rho_0 < \rho_c \Rightarrow \mathcal{E}_{tot} > 0 \Rightarrow$
$R(t) \to \infty$ at $t \to \infty$

(ii) $\rho_0 = \rho_c \Rightarrow \mathcal{E}_{tot} = 0 \Rightarrow$
$R(t) \to \infty$ at $t \to \infty$

} expands forever



(iii) $\rho_0 < \rho_c \Rightarrow \mathcal{E}_{tot} < 0 \Rightarrow R_{max} = \frac{2R_0\rho_0}{\rho_0 - \rho_c}$

bounces back and goes to singularity

General content of the Universe

Let universe contain

- radiation with $p_\gamma = \frac{\rho_\gamma}{3}$
- non-relativistic matter (+ Dark matter, to be discussed later), $p_M = 0$
- cosmological constant

$$p_\Lambda = -\rho_\Lambda$$

How did the universe evolve?

Definition: abundances

$$\Omega_\gamma \equiv \frac{\rho_\gamma}{\rho_{crit}} \; ; \; \Omega_M \equiv \frac{\rho_M}{\rho_{crit}} \; ; \; \Omega_\Lambda \equiv \frac{\rho_\Lambda}{\rho_{crit}}$$

from $H^2 + \frac{k}{R^2} = \frac{8\pi G}{3} \rho_{tot} \Rightarrow$

$$\underbrace{\frac{3H^2}{8\pi G}}_{\rho_c} + \frac{3}{8\pi G} \frac{k}{R^2} = \rho_{tot} \quad \text{or}$$

$$1 = \Omega_\gamma + \Omega_M + \Omega_\Lambda + \Omega_k$$

where $\Omega_k = -\frac{k}{R_0^2 H_0^2}$: curvature contribution

Behaviour of ρ_{tot}:

all defined at t_0 (7)

$$\rho_{tot} = \rho_c \left[\Omega_\Lambda + \Omega_M \left(\frac{R(t_0)}{R(t)} \right)^3 + \Omega_\gamma \left[\frac{R(t_0)}{R(t)} \right]^4 \right]$$

part, associated with cosmological constant does not change

part, associated with matter:

$$\frac{d}{dt} (\rho_M R^3) = 0 \Rightarrow \rho_M \sim \frac{1}{R^3(t)}$$

part, associated with radiation:

$$\frac{d}{dt} (\rho_\gamma R^3) + P_\gamma \frac{d}{dt} R^3 = 0$$

since $P_\gamma = \frac{\rho_\gamma}{3}$, $\frac{d}{dt} (\rho_\gamma R^4) = 0 \Rightarrow$

$$\rho_\gamma \sim \frac{1}{R^4}$$

easy to understand:

# of photons $\sim \frac{1}{R^3}$

energy of photon, $\omega \sim \frac{1}{R}$

Convenient form of Friedmann equation:

$$H^2 = \left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}\rho_{tot} - \frac{K}{R^2} = \frac{8\pi G}{3}\left(\rho_{tot} + \rho_K\right)$$

Last term, to have similar notations.

$$\rho_K = -\frac{K}{R^2}\,\frac{3}{8\pi G} \equiv \rho_c\,\Omega_K\,\frac{R(t_0)^2}{R(t)^2}$$

curvature "abundance"

Ω_K is not independent,

sum rule: $1 = \Omega_\Lambda + \Omega_M + \Omega_\gamma + \Omega_K$

to derive the sum rule, just devide first equation at this page by H^2.

Final form of Friedman equation we will work:

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}\rho_c\left\{\Omega_\Lambda + \Omega_M\left(\frac{R(t_0)}{R}\right)^3 + \Omega_\gamma\left[\frac{R(t_0)}{R(t)}\right]^4 + \Omega_K\left[\frac{R(t_0)}{R(t)}\right]^2\right\} \qquad (****)$$

notation: $R(t_0) = R_0$

This equation can be also written in a form, which has a simple mechanical analogy:

$$\frac{1}{2}\dot{R}^2 + V(R) = 0, \text{ where}$$

$$V(R) = -\frac{4\pi G}{3}\rho_c\left\{\Omega_\Lambda \cdot R^2 + \Omega_M \frac{R_0^3}{R} + \Omega_\gamma \frac{R_0^4}{R^2} + \Omega_K R_0^2\right\}$$

it describes a motion of particle with zero energy in potential $V(R)$

Let us integrate now (****)

notations: $x = \frac{R(t)}{R(t_0)}$; $R(t) = R_0 \cdot x$

$$\Omega_\Lambda + \Omega_M \frac{1}{x^3} + \Omega_K \frac{1}{x^2} + \Omega_\gamma \frac{1}{x^4} \equiv A^2(x)$$

Eq (****) for x:

$$\frac{\dot{x}^2}{x^2} = H_0^2 A^2(x) \Rightarrow \frac{dx}{dt} = H_0 A(x) \cdot x$$

$$dt = \frac{dx}{H_0 x A(x)}; \quad \frac{dt}{dx} = \frac{1}{x H_0 A(x)}$$

This allows us to get immediately from expression on page 2 for $d(z)$:

$$\int_{t_1}^{t_0} \frac{dt}{R(t)} = \int_{t_1}^{t_0} \frac{dt}{dx} \frac{dx}{R_0 x} =$$

$$= \int_{\frac{1}{1+z}}^{1} \frac{dx}{x^2 R_0 H_0 A(x)}$$

t_0 corresponds to $x = 1$

t_1 corresponds to $x = \frac{1}{1+z} = \frac{R(t_1)}{R(t_0)}$

For example, for $k = 0$ we get:

$$d(z) = \frac{1+z}{H_0} \int_{\frac{1}{1+z}}^{1} \frac{dx}{x^2 A(x)}$$

Equation, which can unity all cases:

$$d(z) = \frac{1+z}{H_0 \Omega_k^{1/2}} \sinh \left[\Omega_k^{1/2} \int_{\frac{1}{1+z}}^{1} \frac{dx}{x^2 A(x)} \right]$$

since, e.g.,

for $k = +1$ $R_0^2 H_0^2 = \Omega_k$

taking formally $\Omega_k \to 0$, we get flat case, taking $\Omega_k < 0$ we replace $\sinh \to \sin$

By-product of computation:

age of the Universe.

Take the formula at the bottom of page 9, and integrate it from $t=0$ to t_0

$t=0$ corresponds to $x=0$ (since $R=0$)

$t=t_0$ corresponds to $x=1$

$$t=\frac{1}{H_0}\int_0^1 \frac{dx}{x\,A(x)}$$

How do use $d(z)$ for determination of content of the Universe:

1) measure curve $d(z)$

2) find parameters Ω_γ, Ω_M and Ω_Λ from fitting this curve by expression for $d(z)$

In practice: $\Omega_\gamma \ll \Omega_M$, $\Omega_\gamma \ll \Omega_\Lambda$ and can be neglected

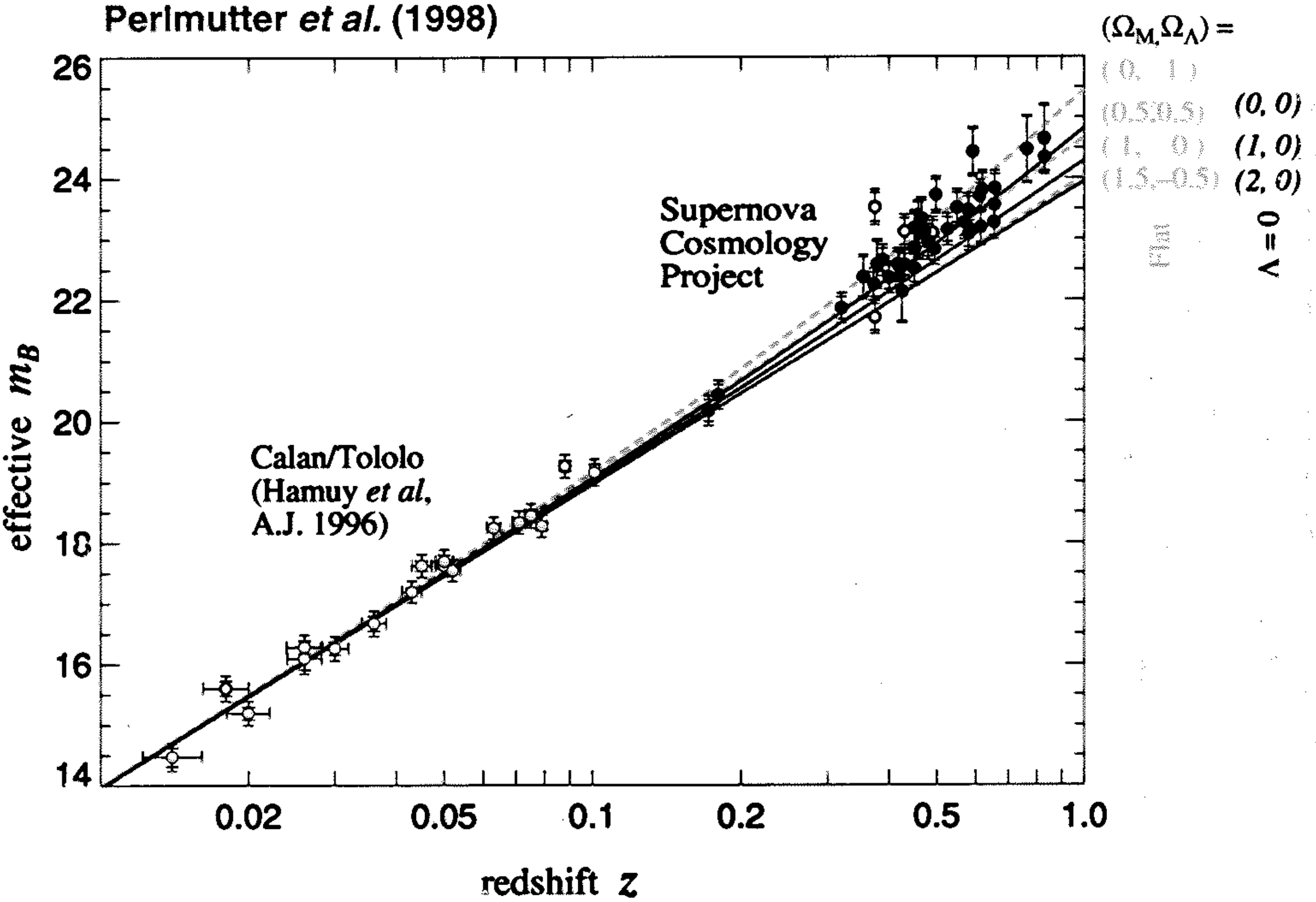
ΛCDM model.

standard candels:

SN1a,

Instrument: Hubble space telescope

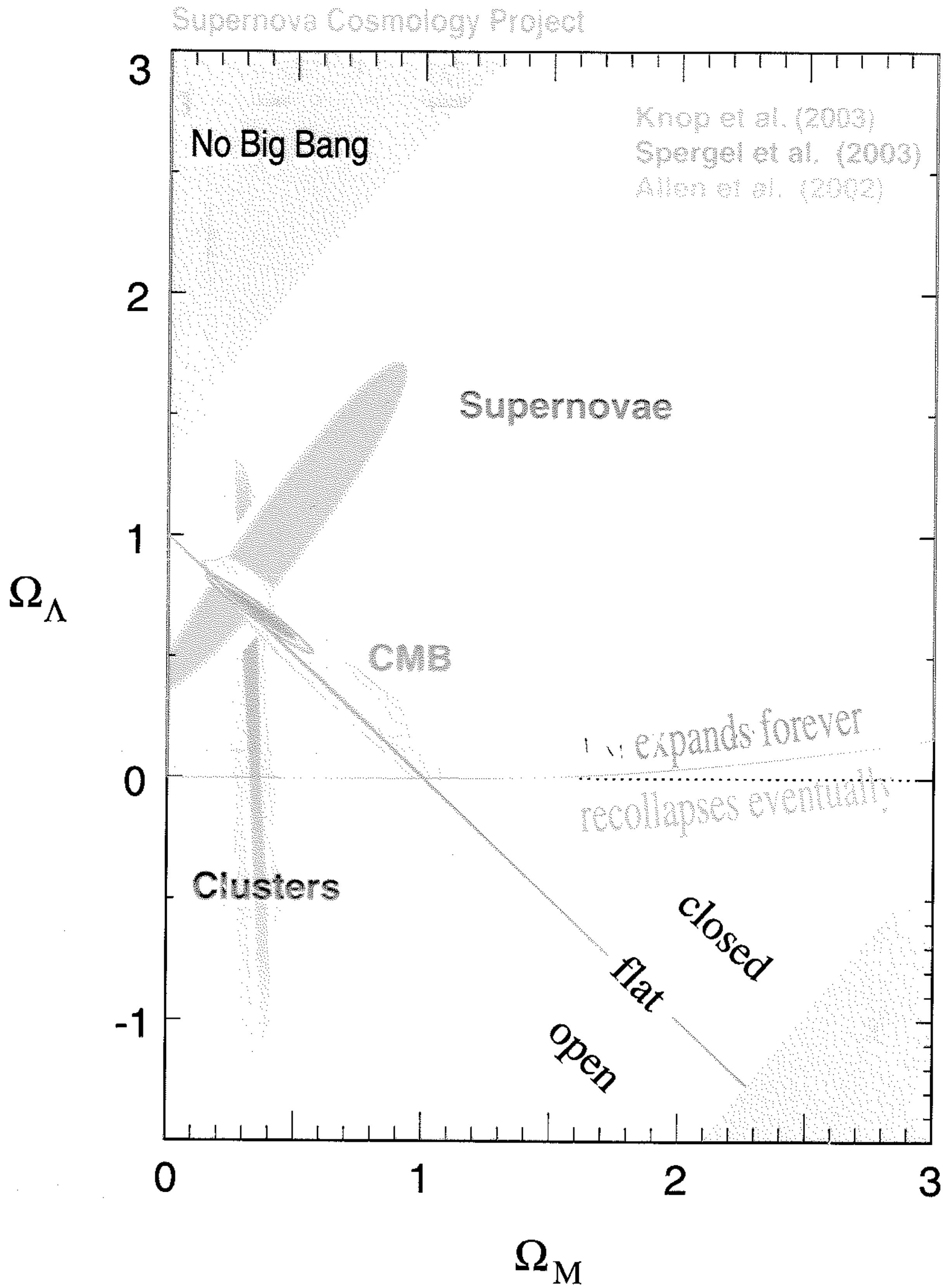
Supernova Cosmology Project
Perlmutter *et al.* (1998)



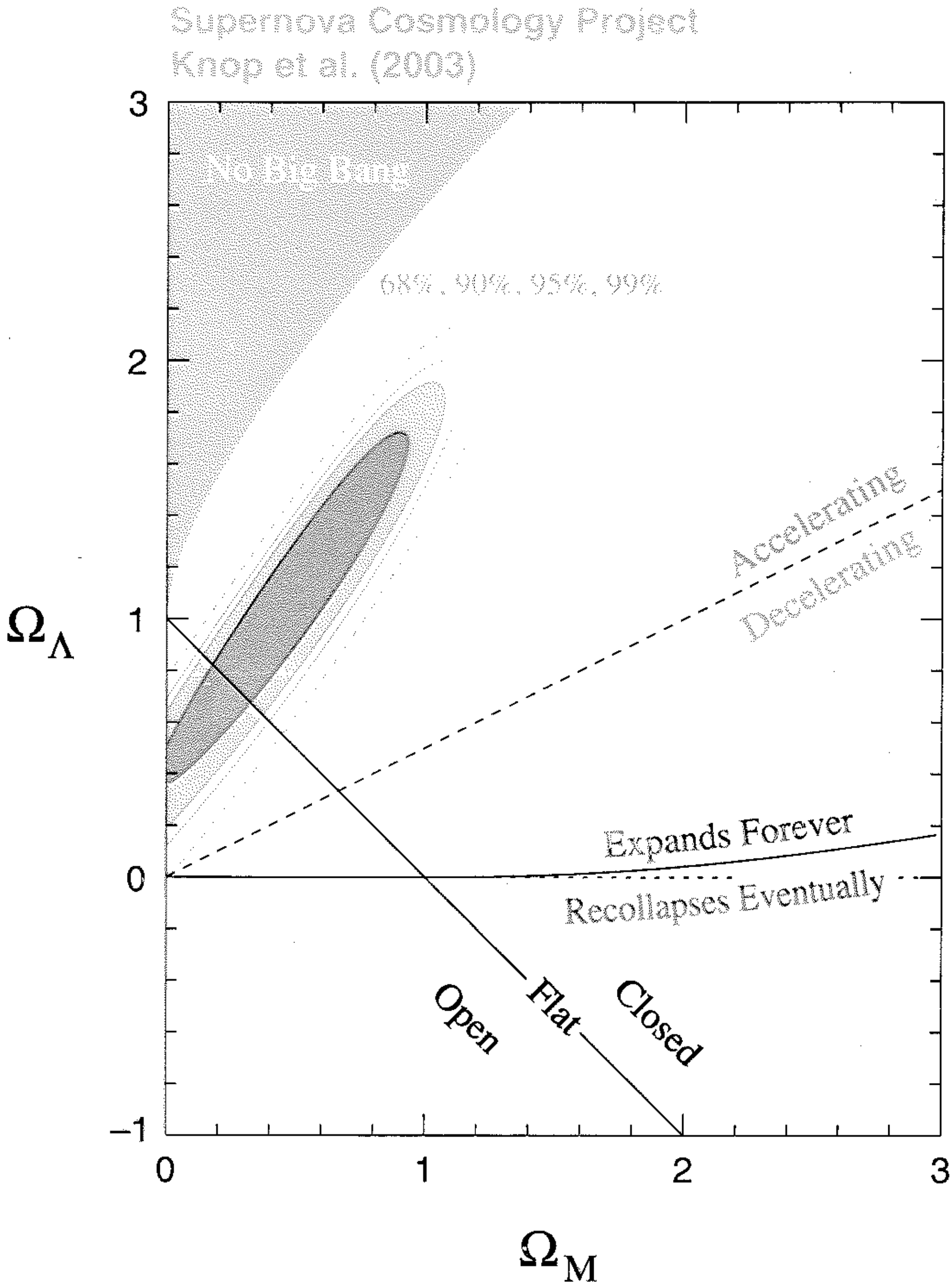
In flat universe: $\Omega_M = 0.28$ [± 0.085 statistical] [± 0.05 systematic]

Prob. of fit to $\Lambda = 0$ universe: 1%

astro-ph/9812133


Supernova Cosmology Project
Knop et al. (2003)
Spergel et al. (2003)
Allen et al. (2002)
No Big Bang
Supernovae
CMB
expands forever
recollapses eventually
Clusters
closed
flat
open
3
2
1
0
-1
0
1
2
3
Ω_Λ
Ω_M

14



Supernova Cosmology Project
Knop et al. (2003)
No Big Bang
68%, 90%, 95%, 99%
Accelerating
Decelerating
Expands Forever
Recollapses Eventually
Open
Flat
Closed
Ω_Λ
Ω_M
3
2
1
0
−1
0
1
2
3

Explanations of page # 10

- Flat: $\Omega_k = 0$
- Closed: $\Omega_k < 0$
- Open: $\Omega_k > 0$
- Recollapses eventually: $R(t) \Rightarrow 0$ for some $t > t_0$ ($\dot{R}(t) = 0$ for some $t > t_0$)
- Expands forever: $\dot{R}(t) > 0$,
- Decelerating: $\ddot{R} < 0$
- accelerating: $\ddot{R} > 0$
- No Big Bang: $R > 0$ for all $t < t_0$ (no singularity)