Relativity and Cosmology II

Lecture # 2

Main points of # 1

- Universe is homogeneous and isotropic at large scales, $\ell \gtrsim 10\,Mpc$
  (Universe visible size : $5 \cdot 10^3\,Mpc$

- constructed 3 isotropic & homogeneous 3d spaces :

$$dl^2 = R^2\left[\frac{d\bar{r}^2}{1-k\bar{r}^2} + \bar{r}^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)\right]$$

$$0 < \varphi \lesssim 2\pi$$

$$0 \leq \theta \leq \pi$$

$$k = \begin{cases} +1 & \text{closed (positive curvature)} \\ 0 & \text{zero curvature} \\ -1 & \text{open (negative curvature)} \end{cases}$$

$0 \leq \bar{r} \leq 1$ for $k = +1$ (finite volume)

$0 \leq \bar{r} < \infty$ for $k = 0, -1$

Plan for today

- metric for cosmology FRW or

Friedmann - Lemaître - Robertson - Walker

Friedmann	Lemaître	Robertson - Walker
1922-1924	1927	1935
first	similar results	proved that it is the only one with spacial isotropy and homogeneity

formulation of Hubble law : Edwin Hubble, 1929

- Friedmann equations
- Friedmann equations without GR
- Einstein static universe
- Expanding universe

Cosmology: Space is always isotropic and homogeneous, but the scale may depend on time.

metric:

$$ds^2 = dt^2 - R^2(t)\, d\bar{l}^2 = g_{\mu\nu}\, dx^\mu dx^\nu$$

Christ. symbols:

$$\Gamma^0_{ij} = \frac{\dot{R}}{R}\frac{1}{R^2}\delta_{ij} \; ; \quad \Gamma^i_{0j} = \frac{\dot{R}}{R}\delta^i_j \; ; \; \Gamma^0_{00} = 0$$

also, $\Gamma^i_{jk} = 0$.

$$R_{00} = -3\,\frac{\ddot{R}}{R}$$

$$R_{ij} = -\left[\frac{\ddot{R}}{R} + 2\frac{\dot{R}^2}{R^2} + \frac{2k}{R^2}\right]\delta_{ij}$$

$$R = -6\left[\ddot{R}/R + \dot{R}^2/R^2 + k/R^2\right]$$

This allows us to write Einstein Equations,

$$G_{\mu\nu} - \lambda g_{\mu\nu} = 8\pi G\, T_{\mu\nu},$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ is Einstein tensor, λ is cosmological constant, $T_{\mu\nu}$ is energy-momentum tensor

The choice of energy-momentum tensor for cosmology:

$$T_{\mu\nu} = (\rho + p)\, u_\mu u_\nu - p\, g_{\mu\nu}$$

p - pressure, ρ - energy density
u - 4-velocity of the medium

relativistic matter equation of state:

$$p = \frac{\rho}{3}$$

non-relativistic matter eq. of state,

$$p = 0$$

Choice of coordinate system: $u_\mu = (1,0,0,0)$,

i.e. $$T_{00} = \rho$$

$$T_{ij} = -p\, g_{ij}$$

Friedmann equations

Einst. equations, 00 component:

$$R_{00} - \frac{1}{2} g_{00} R - \lambda g_{00} = T_{00} \cdot 8\pi G$$

$$-3\frac{\ddot{R}}{R} - \frac{1}{2}\cdot 1 (-6)\left(\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + \frac{k}{R^2}\right) - \lambda$$

$$= \rho \cdot 8\pi G$$

or,

$$\boxed{\frac{\dot{R}^2}{R^2} + \frac{k}{R^2} - \frac{\lambda}{3} = \frac{8\pi G}{3}\rho} \quad (*)$$

Eins. equations, ij components:

$$-\left[\frac{\ddot{R}}{R} + 2\frac{\dot{R}^2}{R^2} + \frac{2k}{R^2}\right] g_{ij} -$$

$$-\frac{1}{2} g_{ij}\left[-6\right]\left[\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + \frac{k}{R^2}\right] - \lambda g_{ij} =$$

$$= 8\pi G \cdot (-p)\, g_{ij}$$

$$\boxed{2\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + \frac{k}{R^2} - \Lambda = -8\pi G \cdot p} \quad (**)$$

All other components, R_{oi} are identically zero

Energy-momentum conservation:

$T^{\mu\nu}{}_{;\nu} = 0$

$$T^{\mu\nu}{}_{;\alpha} = \frac{\partial T^{\mu\nu}}{\partial x^\alpha} + \Gamma^{\mu}_{\alpha\beta} T^{\nu\beta} + \Gamma^{\nu}_{\alpha\beta} T^{\mu\beta};$$

$$T^{\mu\nu}{}_{;\nu} = \frac{\partial T^{\mu\nu}}{\partial x^\nu} + \Gamma^{\mu}_{\nu\beta} T^{\nu\beta} + \Gamma^{\nu}_{\nu\beta} T^{\mu\beta}$$

Let us take

$\mu = 0$:

$$T^{0\nu}{}_{;\nu} = 0 = \frac{\partial T^{0\nu}}{\partial x^\nu} + \Gamma^{0}_{\nu\beta} T^{\nu\beta} + \Gamma^{\nu}_{\nu\beta} T^{0\beta} =$$

$$T^{\mu\nu} = (\rho + p) u^\mu u^\nu - p g^{\mu\nu}$$

$$T^\mu_\nu = [\rho, -p, -p, -p]$$

$$= \frac{\partial T^{00}}{\partial x^0} + \Gamma^0_{00} T^{00} + \Gamma^0_{ij} T^{ij} + \Gamma^\nu_{\nu 0} T^{00} =$$

$$= \frac{\partial \rho}{\partial t} + \frac{\dot{R}}{R} \frac{1}{R^2} \delta_{ij} (-p) g^{ij} + 3 \frac{\dot{R}}{R} \cdot \rho =$$

$$= \frac{\partial \rho}{\partial t} + 3 \frac{\dot{R}}{R} \rho + \frac{\dot{R}}{R} 3p = 0 \qquad (e.c.)$$

or $$\frac{\partial}{\partial t} [\rho R^3] + p \frac{\partial}{\partial t} R^3 = 0$$

or $dE + p\,dV = 0$ [thermod equation
1st law of thermo-
dynamics]

The same result [following from B.
identity]

$$\frac{d}{dt}(*) + 3\frac{\dot{R}}{R}(*) - \frac{\dot{R}}{R}(**) = 0$$

lhs

giving (e.c.)

...ith the use of energy-momentum conservation one can consider different combinations of F. equations:

e.g.

$$\begin{cases} \frac{\dot{R}^2}{R^2} + \frac{k}{R^3} - \frac{\lambda}{3} = \frac{8\pi G}{3}\rho \\ d(\rho R^3) + p\,dR^3 = 0 \end{cases}$$

or

$$\boxed{\frac{\ddot{R}}{R} = \frac{\lambda}{3} - \frac{4\pi G}{3}(\rho + 3p)} \qquad (****)$$

(this is difference between (*) and (**))
nice feature: it does not contain k.

<u>Friedmann equations without general relativity</u>

In the limit

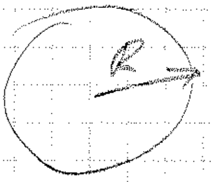
$k = 0$ – flat space

$\lambda = 0$ – „vacuum" does not gravitate

$p = 0$ – non-relativistic dust

One should be able to derive F. equations from Newtonian dynamics

Take a uniform distribution of matter
with energy (mass) density ρ



$$\frac{d}{dt}\left[\frac{4}{3}\pi R^3 \rho\right] = 0 \Rightarrow$$

$$\frac{d}{dt}(\rho R^3) = 0$$ — OK coincides with energy conservation for $p=0$

mass inside sphere does not change

Dynamics:

let m is a test mass $\Rightarrow$

$$m\ddot{R} = \underbrace{-G\cdot\frac{4}{3}\pi R^3 \rho \frac{1}{R^2}\cdot m}_{\text{Newton's law of gravity}}$$

$$\frac{\ddot{R}}{R} = -\frac{4}{3}\pi G\rho$$ — exactly eq (xxxx)

for $\Lambda = 0$, $p = 0$

It works!

Einstein static universe

Original Einstein idea: Universe must be static. If we take $\Lambda = 0$, we get from $(*)$ and $(**)$:

$$\begin{cases} \dfrac{K}{R^2} = +\dfrac{8\pi G}{3}\rho & \text{(i)} \\[2ex] \dfrac{K}{R^2} = -8\pi G\, p & \text{(ii)} \end{cases}$$

Universe is not empty (there are stars!)

$$\rho \neq 0$$

But: stars do not move (static!)

$$p = 0$$

equations (i) and (ii) are not consistent! (i): $K > 0$ (ii) $K = 0$

What to do? Einstein proposal:

$\Lambda \neq 0$ then we have

$$\frac{K}{R^2} - \frac{\Lambda}{3} = \frac{8\pi G}{3}\rho \qquad \text{(i)}$$

$$\frac{K}{R^2} - \Lambda = 0 \qquad \text{(ii)}$$

Consistent solution : Einstein, 1917

$$R = \frac{1}{\sqrt{\lambda}}, \quad k = +1, \quad \lambda = 8\pi G\rho$$

Conceptual problem : take $\rho = 0$

then, if $\lambda \neq 0$, the space is curved?? and non-static ??

indeed, from (*) and (**)

we get

$$\left.\begin{aligned} &\frac{\dot{R}^2}{R^2} + \frac{k}{R^2} - \frac{\lambda}{3} = 0 \\ &2\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + \frac{k}{R^2} - \lambda = 0 \end{aligned}\right]$$

non-trivial solutions for empty space !

after Friedmann solution (1922) appeared, E. said that introducing λ was his „blunder".

Friedmann, 1922

$\lambda = 0$ but universe is not stationary!

equations:

take for simplicity $k = 0$

$$\begin{cases} \dfrac{\dot R^2}{R^2} = \dfrac{8\pi G}{3}\rho \\[2ex] 2\dfrac{\ddot R}{R} + \dfrac{\dot R^2}{R^2} = 0 \qquad \text{(pressure} = 0) \end{cases}$$

Solutions:

$$2\frac{\ddot R}{\dot R} + \frac{\dot R}{R} = 0 \Rightarrow 2\log \dot R + \log R = \text{Const};$$

$$R\dot R^2 = \text{Const}; \quad \dot R = \frac{\text{Const}}{\sqrt{R}}$$

$$\sqrt{R}\, dR = \text{const}\cdot dt;$$

$$R^{3/2} = \text{Const}\cdot t \Rightarrow R = R_0 t^{2/3}$$

$$\frac{\dot R}{R} = \frac{2}{3}\frac{1}{t} \Rightarrow$$

$$\rho = \frac{3}{\underset{2}{8\pi G}}\cdot\frac{4}{\underset{3}{9}}\frac{1}{t^2} = \frac{1}{6\pi G}\frac{1}{t^2}$$

$$\boxed{\begin{aligned} R &= R_0 t^{2/3} \\ \rho &= \frac{1}{6\pi G}\frac{1}{t^2} \end{aligned}}$$

Expanding universe:

prediction - distances between stars must change and increase for this solution!

$$\underbrace{\qquad\qquad}_{\ell} \qquad d\ell^2 = R^2(t)\, d\bar{\ell}^{\,2}$$

$$d\ell = R(t)\, d\bar{\ell}$$

$$\frac{d\ell}{dt} = \dot{R}(t)\, d\bar{\ell}$$

$$\frac{d\ell}{dt} = \frac{\dot{R}(t)}{R(t)}\, d\ell$$

$$\frac{\dot{R}}{R} = \frac{2}{3}\,\frac{1}{t}$$

Was indeed observed later - Hubble

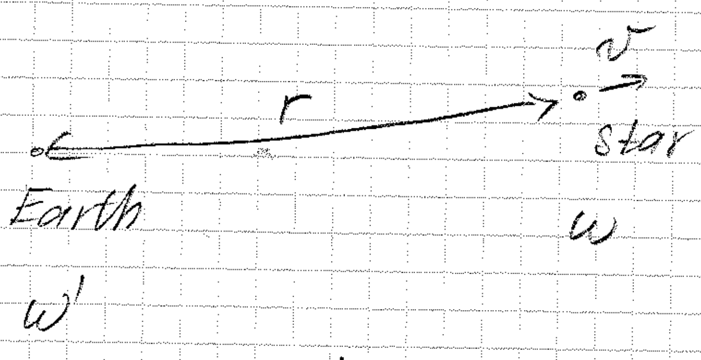
Experimental verification:

(i) find a distance to some star

need a "standard" candle - star with known luminosity

(ii) find the speed of the star

[e.g. by measuring of the shift of a line due to Doppler effect]



$$\omega' = \gamma\omega - \gamma v k; \quad \gamma = \frac{1}{\sqrt{1-v^2}}$$

$$\omega' = \sqrt{\frac{1-v}{1+v}}\,\omega \approx (1-v)\omega$$

Prediction:

$$\boxed{v = Hr}$$

corrected for solar motion. The result, 745 km./sec. for a distance of 1.4×10^6 parsecs, falls between the two previous solutions and indicates a value for K of 530 as against the proposed value, 500 km./sec.

Secondly, the scatter of the individual nebulae can be examined by assuming the relation between distances and velocities as previously determined. Distances can then be calculated from the velocities corrected for solar motion, and absolute magnitudes can be derived from the apparent magnitudes. The results are given in table 2 and may be compared with the distribution of absolute magnitudes among the nebulae in table 1, whose distances are derived from other criteria. N. G. C. 404

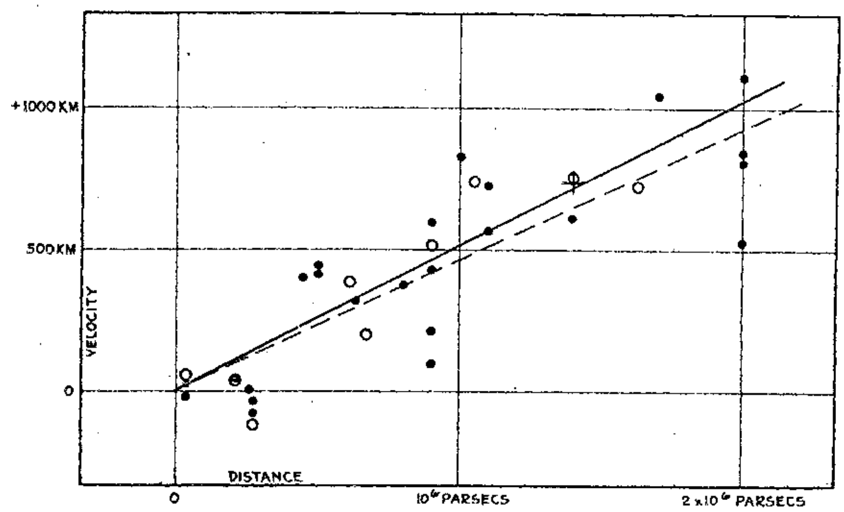


FIGURE 1

Velocity-Distance Relation among Extra-Galactic Nebulae.

Radial velocities, corrected for solar motion, are plotted against distances estimated from involved stars and mean luminosities of nebulae in a cluster. The black discs and full line represent the solution for solar motion using the nebulae individually; the circles and broken line represent the solution combining the nebulae into groups; the cross represents the mean velocity corresponding to the mean distance of 22 nebulae whose distances could not be estimated individually.

can be excluded, since the observed velocity is so small that the peculiar motion must be large in comparison with the distance effect. The object is not necessarily an exception, however, since a distance can be assigned for which the peculiar motion and the absolute magnitude are both within the range previously determined. The two mean magnitudes, -15.3 and -15.5, the ranges, 4.9 and 5.0 mag., and the frequency distributions are closely similar for these two entirely independent sets of data; and even the slight difference in mean magnitudes can be attributed to the selected, very bright, nebulae in the Virgo Cluster. This entirely unforced agreement supports the validity of the velocity-distance relation in a very

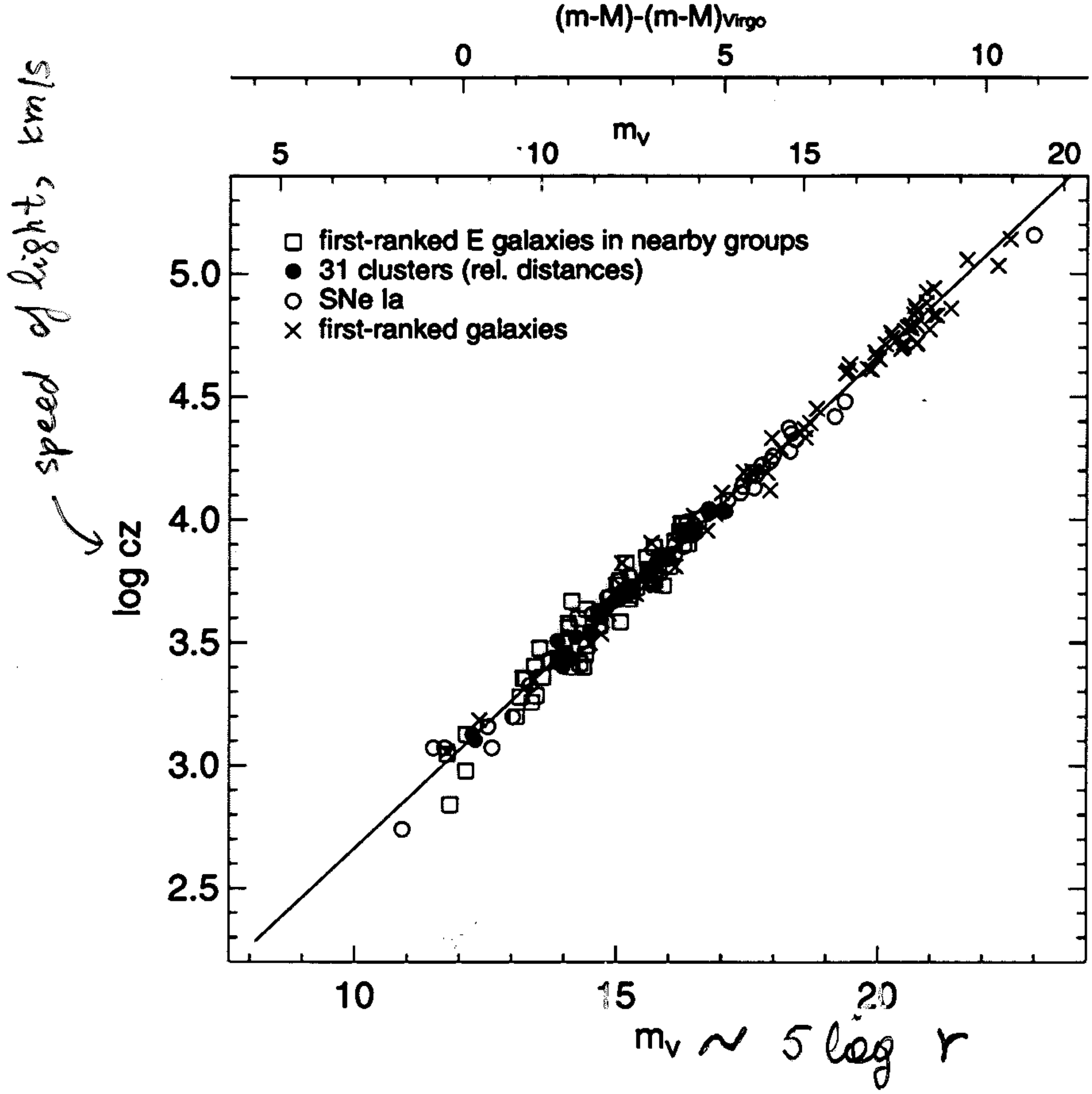


Figure 5: Hubble diagram