Relativity and Cosmology II

Lecture # 1

Introduction:

You can often hear the statements:

- universe is homogeneous
- is isotropic
- was very hot and dense in the past
- contains relic black-body radiation
- contains matter, dark matter,
- contains dark energy
- Light elements like D, ^3He, ^4He, Li were cooked during the first few minutes after the Big Bang
- the structures (clusters of galaxies) were formed from quantum fluctuations
- universe was inflating in the past
- —"— is accelerating now

Aim of these lectures – explain this with the use of GR, equilibrium and non-equilibrium statistical mechanics and particle physics

General plan:

- homogeneous spaces and FRW metric
- Hubble expansion
- Cosmological constant
- Physical processes in the early Univers
  - CMB physics
  - BBN
  - baryogenesis
  - inflation

Today: - properties of Universe at large scales
- FRW metric

Note: notations from Relativity and Cosmology I will be used.

Isotropy & homogeneity of the Universe

Different scales in cosmology & astrophysics

Orders of magnitude:

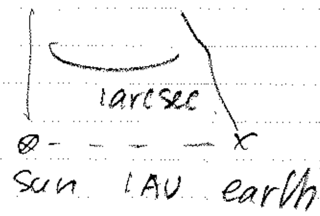
Earth radius $\simeq 6.4 \cdot 10^{8}$ cm

Solar radius $\simeq 7 \cdot 10^{10}$ cm

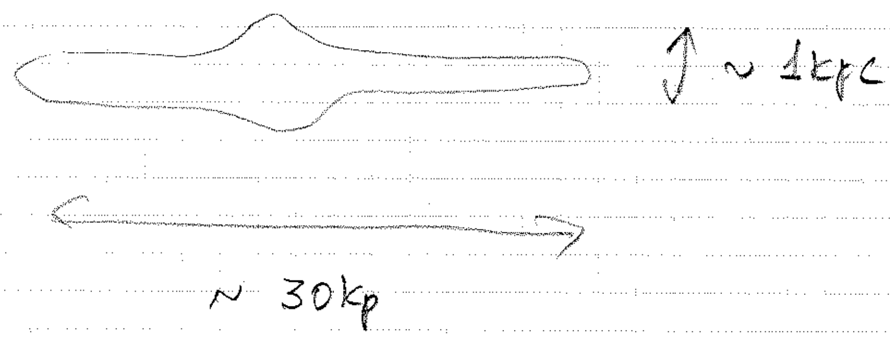
Earth - solar distance $= 1au = 1.5 \cdot 10^{13}$ cm

$pc = 1AU / 1arcsec = 3.26\, ly$

$= 3 \cdot 10^{18}$ cm



Our Galaxy size:



Cluster size: ($\sim 10^{3} - 10^{4}$ galaxies) (Virgo)

~ 10 Mpc

Universe size $\sim 5 \cdot 10^{3}$ Mpc

Universe is isotropic on large scales

large scale ≳ cluster scale.

- To prove- look at different directions of the sky and count the # of galaxies in every direction
- another proof: CMB, see pictures.

Universe is homogeneous ⇒ more difficult to prove -

we cannot go to another point of the universe.

Method: try to find distances between galaxies and reconstruct 3d picture.

Challenge: construct mathematical description of homogeneous & isotropic curved spaces.

Step # 1

forget about time for a moment, will introduce it later.

Space: 3 coordinates x^i

distance: $dl^2 = \gamma_{ij}\, dx^i dx^j$

γ_{ij} : tensor with signature $+++$

The geometry is determined completely by γ_{ij}, and by following from it curvature tensor R_{ijkl} (only spacial indexies in the curvature tensor)

Symmetry properties:

$R_{ijkl} = -R_{jikl}$ (antisymmetric over two first idexies)

$R_{ijkl} = -R_{ijlk}$ two second

$R_{ijkl} = R_{klij}$ symmetry with respect to two first and two second

Step # 2.

Construct an isotropic space in every point

What does it mean?

Take a system of coordinate in which

$$\gamma_{ij} = \delta_{ij} \;, \qquad \underbrace{\partial \gamma_{ij} = 0}$$

Kronekker

Gallilean system of coordinate

(Cartesian system)

Compute in it R_{ijkl} – it must be such that no direction is preffered $\Rightarrow$

$$R_{ijkl} = \text{Const}\, \underbrace{\left[\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} \right]}$$

The only tensor consistent with isotropy.

For a general coordinate system:

$$R_{ijkl} = \zeta \left(\gamma_{ik}\gamma_{jl} - \gamma_{il}\gamma_{jk} \right)$$

Ricci tensor: $R_{ij} = 2\zeta\gamma_{ij}$

scalar curvature: $R = 6\zeta$

Space is homogeneous $\Rightarrow \zeta = \text{Const}$

Three different types of spaces:

$\xi > 0$: constant positive curvature

$\xi = 0$ flat space

$\xi < 0$: constant negative curvature

Metric for $\xi = 0$: $\gamma_{ij} = \delta_{ij}$ - simplest choice

How to define the metric γ for $\lambda > 0$?

take S^3 [3d sphere], defined by equation.

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = R^2$$

This is a 3d space which is obviously homogeneous and isotropic.

distance [the same as in flat 4d space]:

$$dl^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2$$

on a sphere S^3 :

$$x_1 dx_1 + x_2 dx_2 + x_3 dx_3 + x_4 dx_4 = 0 \Rightarrow$$

$$dl^2 = dx_1^2 + dx_2^2 + dx_3^2 + \frac{(x_1 dx_1 + x_2 dx_2 + x_3 dx_3)^2}{R^2 - x_1^2 - x_2^2 - x_3^2}$$

now, introduce coordinates (spherical)

$$x^1 = r\sin\theta\cos\varphi$$
$$x^2 = r\sin\theta\sin\varphi$$
$$x^3 = r\cos\theta$$

Then

$$(dx^1)^2 + (dx^2)^2 + (dx^3)^2 = dr^2 + r^2 d\Omega^2;$$
$$d\Omega^2 = d\theta^2 + \sin^2\theta\, d\varphi^2$$

$$x^1 dx^1 + x^2 dx^2 + x^3 dx^3 = r\,dr$$

So, $$dl^2 = dr^2 + r^2 d\Omega^2 + \frac{r^2 dr^2}{R^2 - r^2} =$$

$$\frac{dr^2}{1 - \frac{r^2}{R^2}} + r^2 d\Omega^2 = R^2\left[\frac{d\bar{r}^2}{1-\bar{r}^2} + \bar{r}^2 d\Omega^2\right]$$

here $\bar{r}$ is dimensionless, and

$$\bar{r} \in [0,1)$$
$$\theta \in [0,\pi]$$
$$\varphi \in [0,2\pi)$$

Scalar curvature $\mathbb{R} = \frac{1}{R^2}$, positive

Another case, negative curvature.

Simply change $R^2 \to -R^2 \Rightarrow$

$$dl^2 = \frac{dr^2}{1 + \frac{r^2}{R^2}} + r^2(\sin^2\theta\, d\varphi^2 + d\theta^2) =$$

$$= R^2\left[\frac{d\bar{r}^2}{1+\bar{r}^2} + \bar{r}^2(\sin^2\theta\, d\varphi^2 + d\theta^2)\right.$$

region for coordinates:

$r \in [0, \infty)$

$\theta \in [0, \pi]$

$\varphi \in [0, 2\pi)$

Equation, which unifies all cases:

$$dl^2 = R^2\left[\frac{d\bar{r}^2}{1 - k\bar{r}^2} + \bar{r}^2 d\Omega^2\right]$$

$k = +1$ – closed, positive curvature

$k = 0$ – flat, 0 —"—

$k = -1$ – open, negative —"—

Remark

Embedding of a space with negative curvature to euclidean spaces:

2d (Lobachevski) space:

in 3d - impossible

5d, 6d - possible

4d - not known (locally - yes, globally - not known)

Gromov: hyperbolic k - space can be embedded

in 5k-5 Euclidean space

$k=3 \Rightarrow \underline{10}$

Cartan: $\geq 2k-1 = 7$