Relativity and Cosmology II

Lecture #1

Introduction:

You can often hear the statements:

- universe is homogeneous
- universe is isotropic
- was very hot and dense in the past
- contains relic black-body radiation
- contains matter, dark matter, contains dark energy
- light elements like D, He, 3He, 4He, Li were cooked during the first few minutes after the Big Bang
- the structures (clusters of galaxies) were formed from quantum fluctuations
- universe was inflating in the past
- is accelerating now

Aim of these lectures - explain this with the use of GR, equilibrium and non-equilibrium statistical mechanics, and particle physics.
General plan:

- homogeneous spaces and FRW metric
- Hubble expansion
- Cosmological constant
- Physical processes in the early Universe
  - CMB physics
  - BBN
  - baryogenesis
  - inflation

Today:
- properties of Universe at large scales
- FRW metric

Note: notations from Relativity and Cosmology I will be used.
Isotropy & homogeneity of the Universe.

Different scales in cosmology & astrophysics

Order of magnitudes:

Earth radius = \(6.4 \times 10^8\) cm

Solar radius = \(7 \times 10^9\) cm

Earth–solar distance = 1 AU = \(1.5 \times 10^{13}\) cm

\[
p_c = \frac{1 \text{AU}}{1 \text{arcsec}} = 3.26 \text{ly} = 3 \times 10^{18}\text{cm}
\]

Our Galaxy size:

\(\sim 1\text{kpc}\)

\(\sim 30\text{kpc}\)

Cluster size: \(\sim 10^3 - 10^6\) galaxies \((\text{VIRGO})\)

\(\sim 10\text{Mpc}\)

Universe size \(\sim 10^3\text{Mpc}\)
The universe is isotropic on large scales.

Large scale = cluster scale.

- To prove: look at different directions of the sky and count the # of galaxies in every direction.

- Another proof: CMB, see pictures.

Universe is homogeneous ⇒ more difficult to prove - we cannot go to another point of the universe.

Method: try to find distances between galaxies and reconstruct 3d picture.

Challenge: construct description of homogeneous & isotropic curved space.
Step # 1

For the moment, will introduce it later.

Space: 3 coordinates $x^i$

distance: $dl^2 = g_{ij} dx^i dx^j$

$g_{ij}$: tensor with signature +++

The geometry is determined completely by $g_{ij}$, and by following from it curvature tensor $R_{ijke}$ (only special indices in the curvature tensor).

Symmetry properties:

$R_{ijke} = -R_{jiei}$ (antisymmetric over two first indices)

$R_{ijke} = -R_{ijek}$ two second

$R_{ijke} = R_{keli}$ symmetry with respect to two first and two second
Construct an isotropic space in every point

What does it mean?

Take a system of coordinate in which

\[ \delta_{ij} = \delta_{ij}, \quad \delta_{ij} = 0 \]

Kronecker

Galilean system of coordinate
(Cartesian system)

Compute in it \( R_{jke} \) — it must be such that no direction is preferred:

\[ R_{jke} = \text{const} \left[ \delta_{ik} \delta_{je} - \delta_{ie} \delta_{jk} \right] \]

The only tensor consistent with isotropy.

For a general coordinate system:

\[ R_{jke} = \gamma \left( \delta_{ik} \delta_{je} - \delta_{ie} \delta_{jk} \right) \]

Ricci tensor:

\[ R_{ij} = \gamma \gamma_{ij} \]

Scalar curvature:

\[ R = 6 \gamma \]

Space is homogeneous \( \Rightarrow \gamma = \text{const} \)
Three different types of spaces:

\( \xi > 0 \): constant positive curvature

\( \xi = 0 \) flat space

\( \xi < 0 \): constant negative curvature

Metric for \( \xi = 0 \): \( g_{ij} = \delta_{ij} \) - simplest choice

How to define the metric \( g \) for \( \xi > 0 \)?

Take \( S^3 \) [3d sphere], defined by equation

\[ x_1^2 + x_2^2 + x_3^2 + x_4^2 = R^2 \]

This is a 3d space which is obviously homogeneous and isotropic.

Distance [the same as in flat 4d space]:

\[ ds^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 \]

on a sphere \( S^3 \):

\[ x_1^2 + x_2^2 + x_3^2 + x_4^2 = R^2 \Rightarrow \]

\[ ds^2 = dx_1^2 + dx_2^2 + dx_3^2 + \left( x_1^2 + x_2^2 + x_3^2 \right)^2 \]

\[ R^2 = x_1^2 + x_2^2 + x_3^2 \]
Scalar curvature $R = \frac{d}{dr} \nabla r$.

Here $\nabla r$ is dimensionless, and

$$\frac{d}{dr} + r \frac{d}{dr} = 2r \frac{d}{dr} + r \frac{d}{dr} = 2r \frac{d}{dr} + 2r \frac{d}{dr} = 4r \frac{d}{dr}$$

So,

$$d_{d} = dr^2 + r^2 d\Omega^2 = 2r^2 \frac{d}{dr} + 4r^2 \frac{d}{dr}$$

$$d_{d} = dr^2 + r^2 \sin^2 \theta d\phi^2$$

$$d_{d} = dr^2 + r^2 d\Omega^2$$

There

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$d_{d} = dr^2 + r^2 d\Omega^2$$

$$d_{d} = dr^2 + r^2 d\Omega^2$$
Another case, negative curvature.

Simply change \( R^2 \Rightarrow -R^2 \Rightarrow \)

\[ ds^2 = \frac{dr^2}{1 + \frac{r^2}{R^2}} + r^2 (\sin^2 \theta d\phi^2 + d\theta^2) = R^2 \left[ \frac{d\vec{r}^2}{1 + \frac{r^2}{R^2}} + R^2 (\sin^2 \theta d\phi^2 + d\theta^2) \right] \]

region for coordinates:

\( r \in [0, \infty) \)

\( \theta \in [0, \pi] \)

\( \phi \in [0, 2\pi] \)

which unifies all cases:

\[ ds^2 = R^2 \left[ \frac{d\vec{r}^2}{1 - \kappa \frac{r^2}{R^2}} + R^2 d\Omega^2 \right] \]

\( \kappa = +1 - \text{closed, positive curvature} \)

\( \kappa = 0 - \text{flat, 0} \)

\( \kappa = -1 - \text{open, negative} \)
Remark (to Euclidean spaces)
Embedding of a space with negative curvature:

2d (Lobachevsky) space:
in 3d - impossible
5d, 6d - possible

4d - not known (locally - yes, globally - not known)

hyperbolic

Gromov: $K$-space can be embedded

in 5x5 Euclidean space

$k = 3 \Rightarrow 10$

Cartan: $2k - 1 = 7$