# Quantum Field Theory

## Homework 2: solutions

### Exercise 1

Given the Lorentz transformation:

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu},$$
$$\psi'_L(x') = \Lambda_L \psi_L(\Lambda^{-1}x') = e^{-\frac{1}{2}(i\theta^i+\eta^i)\sigma^i}\psi_L(\Lambda^{-1}x'),$$

where $\theta$, $\eta$ are the parameters associated respectively to rotations and boosts, one can consider the bilinear $\psi_L^{\dagger}\bar{\sigma}^{\mu}\psi_L$, where as usual $\bar{\sigma}^{\mu} = (1, -\sigma^i)$. We recall the notation for spinorial indices:

$$\psi_{L\,\alpha}, \qquad \psi^{\dagger}_{L\,\dot{\beta}}, \qquad (\bar{\sigma}^{\mu})^{\dot{\beta}\alpha},$$

since the † transforms undotted indexes in dotted ones and vice versa. Thus the transformation properties of the bilinear are:

$$\psi'^{\dagger}_{L\,\dot{\beta}}(\bar{\sigma}^{\mu})^{\dot{\beta}\alpha}\psi'_{L\,\alpha} = \psi^{\dagger}_{L\,\dot{\gamma}}(\Lambda_L^{\dagger})^{\dot{\gamma}}_{\dot{\beta}}(\bar{\sigma}^{\mu})^{\dot{\beta}\alpha}(\Lambda_L)^{\delta}_{\alpha}\psi_{L\,\delta}.$$

We now recall that $\Lambda_L^{\dagger}\bar{\sigma}^{\mu}\Lambda_L = \Lambda^{\mu}{}_{\nu}\bar{\sigma}^{\nu}$ to conclude that this bilinear transforms in the representation (1/2, 1/2). This is the expected results since all the spinorial indices are contracted while one vector index is free (in practice the $\bar{\sigma}^{\mu}$ represent the Clebsch-Gordan coefficient needed to pass from the $(1/2, 0) \otimes (0, 1/2)$ to the (1/2, 1/2)). Let us consider now the left doublet $\Psi_L$ and the right singlet $\psi_R$ with transformation properties under $SU(2)$ and Lorentz given by:

$$\text{Lorentz}\begin{cases} x'^{\mu} = \Lambda^{\mu}{}_{\nu}x^{\nu} \\ (\Psi'_L)^a_{\alpha}(x') = (\Lambda_L)_{\alpha}^{\ \beta}(\Psi_L)^a_{\beta}(\Lambda^{-1}x') \\ (\psi'_R)^{\dot{\beta}}(x') = (\Lambda_R)^{\dot{\beta}}_{\ \dot{\delta}}(\psi_R)^{\dot{\delta}}(\Lambda^{-1}x') \end{cases}$$

$$\text{Isospin}\begin{cases} x'^{\mu} = x^{\mu} \\ (\Psi'_L)^a_{\alpha}(x') = U^a_{\ b}(\Psi_L)^b_{\alpha}(x) \\ (\psi'_R)^{\dot{\beta}}(x') = (\psi_R)^{\dot{\beta}}(x) \end{cases}$$

Hence the bilinear $\psi_R^{\dagger}\Psi_L = \psi_R^{\dagger\,\alpha}\Psi_{L\,\alpha}$ transforms as:

$$\text{Lorentz:}\qquad \psi'^{\dagger\,\alpha}_R\Psi'_{L\,\alpha} = (\Lambda_R\psi_R)^{\dagger\,\alpha}(\Lambda_L\Psi_L)_{\alpha} = \psi_R^{\dagger\gamma}(\Lambda_R^{\dagger})_{\gamma}^{\ \alpha}(\Lambda_L)_{\alpha}^{\ \beta}\Psi_{L\,\beta} = \psi_R^{\dagger\,\gamma}\Psi_{L\,\gamma},$$

since $\Lambda_R^{\dagger} = \Lambda_R^{-1}$. Hence the latter is a scalar under Lorentz transformations. Note that we have omitted the $x$ dependence but clearly the complete relation would be:

$$\psi'^{\dagger}_R(x')\Psi'_L(x') = \psi^{\dagger}_R(\Lambda^{-1}x')\Psi_L(\Lambda^{-1}x'),$$

which is the usual one for scalar quantities. Under $SU(2)$ transformation one gets:

$$\text{Isospin:}\qquad \psi'^{\dagger}_R\Psi'^a_L = \psi^{\dagger}_R U^a_{\ b}\Psi^b_L = U^a_{\ b}\psi^{\dagger}_R\Psi^b_L.$$

therefore the bilinear is an Isospin doublet. Still, this was expected since the spinor indices are all contracted while, concerning $SU(2)$, we are considering the product: $0 \otimes 1/2 = 1/2$.

Let's consider finally the term $\Psi_L^{\dagger}\sigma^i\,\bar{\partial\!\!\!/}\Psi_L$:

$$(\Psi_L^{\dagger})_{\dot{\alpha}\,b}(\sigma^i)^b{}_a(\bar{\sigma}^{\mu})^{\dot{\alpha}\beta}\partial_{\mu}(\Psi_L)^a_{\beta},$$

where by convention the † exchanges dotted with undotted indices and lowers the Isospin index. Then the two transformations give:

$$\begin{aligned}
\text{Lorentz:}\quad & (\Psi'^{\dagger}_L)_{\dot\alpha\, b}(\sigma^i)^b{}_a(\bar\sigma^\mu)^{\dot\alpha\beta}\partial'_\mu(\Psi'_L)^a_\beta = (\Psi^\dagger_L)_{\dot\gamma}(\Lambda^\dagger_L)^{\dot\gamma}{}_{\dot\alpha}\sigma^i(\bar\sigma^\mu)^{\dot\alpha\beta}\Lambda_\mu{}^\nu\partial_\nu(\Lambda_L)_\beta^{\ \delta}(\Psi_L)_\delta = \Lambda^\mu{}_\nu\Lambda_\mu{}^\rho\Psi^\dagger_L\sigma^i\bar\sigma^\nu\partial_\rho\Psi_L \\
& = (\Lambda^{-1})_\nu{}^\mu\Lambda_\mu{}^\rho\Psi^\dagger_L\sigma^i\bar\sigma^\nu\partial_\rho\Psi_L = \Psi^\dagger_L\sigma^i\bar\sigma^\nu\partial_\nu\Psi_L, \\
\text{Isospin:}\quad & (\Psi'^{\dagger}_L)_b(\sigma^i)^b{}_a\,\bar{\partial\!\!\!/}(\Psi'_L)^a = (\Psi^\dagger_L)_c(U^\dagger)^c{}_b(\sigma^i)^b{}_a\,\bar{\partial\!\!\!/}U^a{}_d(\Psi_L)^d = R^{(j=1)}[U]^i{}_j(\Psi^\dagger_L)_b(\sigma^j)^b{}_a\,\bar{\partial\!\!\!/}(\Psi_L)^a,
\end{aligned}$$

where we have made use of the relations:

$$\begin{aligned}
&(\Lambda^\dagger_L)^{\dot\gamma}{}_{\dot\alpha}(\bar\sigma^\mu)^{\dot\alpha\beta}(\Lambda_L)_\beta^{\ \delta} = \Lambda^\mu{}_\nu(\bar\sigma^\mu)^{\dot\gamma\delta}, \\
&(U^\dagger)^c{}_b(\sigma^i)^b{}_aU^a{}_d = R^{(j=1)}[U]^i{}_j(\sigma^j)^c{}_d.
\end{aligned}$$

In the end the latter bilinear is a Lorentz scalar while is an Isospin vector (that is to say it transforms in the $j = 1$ representation of $SU(2)$).

## Exercise 2

Given the left doublet spinor $\Psi_L$, the two right singlets spinors $u_R$, $d_R$ and the scalar doublet $\Phi$ one wants to construct the most general terms of dimension equal or less than 4, containing at least one scalar and one fermion. Firstly we notice that the dimension in energy of scalar and spinors is not the same:

$$\begin{aligned}
&[\partial_\mu\Phi\partial^\mu\Phi] = E^4 \quad\Longrightarrow\quad [\Phi] = E \\
&[\psi^\dagger\,\bar{\partial\!\!\!/}\psi] = E^4 \quad\Longrightarrow\quad [\psi] = E^{3/2}.
\end{aligned}$$

Clearly a term containing only one spinor cannot be invariant under Lorentz transformations. Therefore at least two spinors have to be present. Since it's asked the presence of the scalar as well, the minimal dimension of a Lorentz invariant term with the required field content is $(E^{3/2})^2 \cdot E = E^4$; adding a futher derivative or fields would increase the dimension to 5 or more.
Secondly, let's consider the Lorentz invariant built with two spinors and no derivatives:

$$u^\dagger_R\Psi_L, \qquad d^\dagger_R\Psi_L \qquad + \text{ their h.c.}$$

Indeed terms containing only left or only right spinors, like $\Psi^\dagger_L\Psi_L$ or $u^\dagger_Ru_R$, are not invariant. The kinetic term $\Psi^\dagger_L\,\bar{\partial\!\!\!/}\Psi_L$ is infact invariant but, as already discussed, adding scalar field we would get quantities with dimension higher than 4. The two terms listed above are Lorentz scalars but are Isospin doublets (see the previous exercise). In order to obtain Isospin invariant one has to contract them with some Lorentz scalar carrying an $SU(2)$ index. We can make use of the scalar doublet $\Phi$; recalling the form of the kinetic term of an Isospin doublet of Lorentz scalars, $\partial_\mu\Phi^\dagger\partial^\mu\Phi$, the first attempt is to build:

$$\Phi^{\dagger a}(u^\dagger_R)^\alpha(\Psi_L)_{\alpha\, a} \qquad \Phi^{\dagger a}(d^\dagger_R)^\alpha(\Psi_L)_{\alpha\, a} \qquad + \text{ their c.c.}$$

These are scalars under both Lorentz and Isospin transformation. One needs to check the invariance under $U(1)$:

$$\begin{aligned}
&\Phi'^{\dagger a}(u'^\dagger_R)^\alpha(\Psi'_L)_{\alpha\, a} = e^{i(-\frac{1}{2}-\frac{2}{3}+\frac{1}{6})\alpha}\Phi^{\dagger a}(u^\dagger_R)^\alpha(\Psi_L)_{\alpha\, a} = e^{-i\alpha}\Phi^{\dagger a}(u^\dagger_R)^\alpha(\Psi_L)_{\alpha\, a}, \\
&\Phi'^{\dagger a}(d'^\dagger_R)^\alpha(\Psi'_L)_{\alpha\, a} = e^{i(-\frac{1}{2}+\frac{1}{3}+\frac{1}{6})\alpha}\Phi^{\dagger a}(d^\dagger_R)^\alpha(\Psi_L)_{\alpha\, a} = \Phi^{\dagger a}(d^\dagger_R)^\alpha(\Psi_L)_{\alpha\, a}.
\end{aligned}$$

Only the second term is invariant under all the symmetry groups.
The second term which is asked for is subtler; one could notice that the quantity $\Phi_a(u^\dagger_R)^\alpha(\Psi_L)_{\alpha\, b}$ is a Lorentz and $U(1)$ invariant. At this stage however this term is a $SU(2)$ tensor transforming as:

$$\Phi'_a(u'^\dagger_R)^\alpha(\Psi'_L)_{\alpha\, b} = U_a{}^cU_b{}^d\Phi_c(u^\dagger_R)^\alpha(\Psi_L)_{\alpha\, d}.$$

If one were able to contract the free indices $a$ and $b$ with a tensor $T^{ab}$ such that $U_a{}^cU_b{}^dT^{ab} = T^{cd}$ then he would obtain a Isospin invariant. One can easily verify that the $\delta^{ab}$ is not a good candidate (also because in the present notation the delta has indices $\delta^b_a$). However one can try with the antisymmetric tensor $\varepsilon^{ab}$:

$$U_a{}^cU_b{}^d\varepsilon^{ab} = \det(U)\varepsilon^{cd} = \varepsilon^{cd}$$

since $\det(U) = 1$. This formula can be proved considering that the l.h.s. is antisymmetric in $c, d$, therefore has to be proportional to the only antisymmetric tenson in two dimension: $\varepsilon$. To obtain the coefficient we contract both right and left members with $\varepsilon_{cd}$:

$$\varepsilon^{cd}\varepsilon_{cd} = 2 = \varepsilon_{cd}{U_a}^c{U_b}^d\varepsilon^{ab} = 2({U_1}^1{U_2}^2 - {U_1}^2{U_1}^2) = 2\det(U).$$

In the end the term invariant under Lorentz, isospin and $U(1)$ transformation is

$$\varepsilon^{cd}\Phi_c(u_R^\dagger)^\alpha(\Psi_L)_{\alpha\, d}.$$