Quantum Field Theory

Homework Set 1

Solve the set of exercises and give it back on Tuesday 10th December. The correction will be provided the following week. No marks will be assigned (neither positive nor negative): this set is thought to test your level of understanding and make you conscious of your preparation and needs. In order to make the correction faster you are kindly asked to follow the notation adopted during the lectures and the exercises.

Exercise 1: Scale transformations

The action of a free real massless scalar field in d dimension reads:

$$\mathcal{S} = \int dt d^{d-1}x \; \mathcal{L}(t,x), \qquad \mathcal{L} = \frac{1}{2}\partial^\mu \phi \partial_\mu \phi,$$

where $\mu = 0,1,..,d-1$, $x^0 = t$ and the indices are raised and lowered with the d-dimensional metric $\eta_{\mu\nu} = \mathrm{diag}(1,-1,..,-1)$.
Consider the transformation

$$\begin{aligned} x^\mu &\longrightarrow x'^\mu = e^\lambda x^\mu \\ \phi(x) &\longrightarrow \phi'(x') = e^{k\lambda}\phi(e^{-\lambda}x') \end{aligned} \tag{1}$$

where $\lambda \in \mathbb{R}$.

- Compute the value of k such that the above transformation defines a symmetry of the theory.
- For $d = 4$ compute the energy momentum tensor $T^\mu_{\ \nu}$ of the theory. Compute the trace of the energy momentum tensor.
- Consider the *improved energy momentum tensor* $K^\mu_{\ \nu} = T^\mu_{\ \nu} + A\delta^\mu_\nu \Box \phi^2 + B\partial^\mu\partial_\nu\phi^2$. Show in $d = 4$ that we can choose the values of A and B in such a way that $\partial_\mu K^\mu_{\ \nu} = 0$ and $K^\mu_{\ \mu} = 0$.
- For $d = 4$ compute the Noether's current S^μ associated to the symmetry defined in (1). Express it in terms of the improved energy momentum tensor $K^\mu_{\ \nu}$ and show which constraints $\partial_\mu S^\mu = 0$ imposes on $K^\mu_{\ \nu}$.
- Find for which values of d the addition of the following potentials to the free Lagrangian density doesn't spoil the symmetry (that is to say the transformation is still a symmetry of the new Lagrangian density):

$$\mathcal{L}' = \partial^\mu\phi\partial_\mu\phi - \begin{cases} \frac{m^2}{2!}\phi^2, \\ \text{or} \\ \frac{\beta}{3!}\phi^3, \\ \text{or} \\ \frac{\alpha}{4!}\phi^4. \end{cases}$$

  and compute the dimension (in powers of energy) of the parameters m, β, α for those values of d.
- Discuss this result.

Exercise 2: Baryon classification

Consider the Isospin doublet $q = \binom{u}{d}$, that is to say a state transforming in the representation $j = 1/2$ of $SU(2)$. We denote $|\uparrow\rangle = u$, $|\downarrow\rangle = d$, the two eigenvectors of the third generator of $SU(2)$. These states correspond to the *up quark* and *down quark* respectively.

- Consider the Baryons (bound states of *three* quarks) that one can obtain starting from the doublet q and classify them in terms of their Isospin (representation of $SU(2)$ they belong to).

Exercise 3: Charges Algebra

Consider a Lie symmetry group $\mathcal{G}$ described by parameters $\{\alpha^i\}$, acting on coordinates and fields as

$$\begin{aligned} g: &\qquad x^\mu \longrightarrow x'^\mu = x^\mu, \\ g: &\qquad \phi_a(x) \longrightarrow \phi'_a(x') = \mathcal{R}(g)_a{}^b \phi_b(x), \end{aligned}$$

where $\mathcal{R}(g)_a{}^b$ is the representation of an element $g \in \mathcal{G}$.
Show that the Noether's charges Q_i together with the product defined by the Poisson brackets form an Algebra which is isomorphic to the Lie Algebra of $\mathcal{G}$.