Quantum Field Theory

Homework 1: solutions

Exercise 1

The action for a free massless scalar field in d dimension is

$$\mathcal{S} = \frac{1}{2}\int dt d^{d-1}x\ \partial_\mu\phi(x)\partial^\mu\phi(x).$$

We consider the scale transformation labelled by the parameter $\lambda \in \mathbb{R}$ and defined as

$$\begin{aligned}
&x'^\mu = e^\lambda x^\mu,\\
&\phi'(x') = e^{k\lambda}\phi(x) = e^{k\lambda}\phi(e^{-\lambda}x').
\end{aligned}$$

Expanding for infinitesimal parameter one gets

$$\begin{aligned}
&x'^\mu \simeq x^\mu + \lambda x^\mu + O(\lambda^2) \implies \epsilon^\mu = -x^\mu,\\
&\phi'(x) = (1 + k\lambda + O(\lambda^2))\phi(x - \lambda x + ..) \simeq (1 + k\lambda + O(\lambda^2))(\phi(x) - \lambda x^\mu\partial_\mu\phi(x) + O(\lambda^2))\\
&\qquad \simeq \phi(x) + k\lambda\phi(x) - \lambda x^\mu\partial_\mu\phi(x) + O(\lambda^2) \implies \Delta(x) = k\phi(x) - x^\mu\partial_\mu\phi(x).
\end{aligned}$$

Here the usual indices a, i labeling different fields and parameters have disappeared since they assume only the value $a = i = 1$. In order to define a symmetry of the theory these transformation must leave invariant the action:

$$\begin{aligned}
&x' = e^\lambda x \qquad d^dx' = e^{d\lambda}d^dx \qquad \partial'_\mu = e^{-\lambda}\partial_\mu,\\
&\partial'_\mu\phi'(x') = e^{k\lambda}\partial'_\mu\phi(x) = e^{(k-1)\lambda}\partial_\mu\phi(x),\\
&\frac{1}{2}\int d^dx\partial_\mu\phi(x)\partial^\mu\phi(x) \longrightarrow \frac{1}{2}\int d^dx\partial_\mu\phi(x)\partial^\mu\phi(x)e^{(2k-2+d)\lambda}\\
&\implies (2k - 2 + d)\lambda = 0 \implies k = 1 - \frac{d}{2}.
\end{aligned}$$

In last equation we have discarded the solution $\lambda = 0$, which corresponds to the identical transformation, which is always an uninteresting symmetry.
In four dimension, $k = -1$ and the Noether's current reads

$$\begin{aligned}
S^\mu &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta - \epsilon^\mu\mathcal{L} = -\left(\phi + x^\nu\partial_\nu\phi\right)\partial^\mu\phi + \frac{1}{2}x^\mu\left(\partial_\nu\phi(x)\partial^\nu\phi(x)\right)\\
&= -\phi\partial^\mu\phi - x^\nu\partial_\nu\phi\partial^\mu\phi + \frac{1}{2}x^\mu\partial_\nu\phi\partial^\nu\phi.
\end{aligned}$$

Recalling the definition of the energy momentum tensor associated to this Lagrangian

$$T^\mu_{\ \rho} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\rho\phi - \delta^\mu_\rho\mathcal{L} = \partial_\rho\phi\partial^\mu\phi - \frac{1}{2}\delta^\mu_\rho\left(\partial_\nu\phi\partial^\nu\phi\right),$$

one has

$$T^\mu_{\ \mu} = \partial_\mu\phi\partial^\mu\phi - \frac{4}{2}\partial_\nu\phi\partial^\nu\phi = -\partial_\nu\phi\partial^\nu\phi\ .$$

One can consider an *improved energy momentum tensor* $K^\mu_{\ \rho}$ adding the terms

$$K^\mu_{\ \rho} = T^\mu_{\ \rho} + A\delta^\mu_\rho\Box\phi^2 + B\partial_\rho\partial^\mu\phi^2.$$

The choice of the constant A, B is fixed by the requirement that the above expression be conserved (as the original energy-momentum tensor) and in addition traceless:

$$\begin{aligned}
&\partial_\mu K^\mu_{\ \rho} = \partial_\mu T^\mu_{\ \rho} + (A + B)\partial_\rho\Box\phi^2 = 0 \implies A + B = 0\,,\\
&K^\mu_{\ \mu} = T^\mu_{\ \mu} + 4A\Box\phi^2 - A\Box\phi^2 = 0.
\end{aligned}$$

Using the identity

$$\Box\phi^2 = 2\partial_\mu\phi\partial^\mu\phi + 2\phi\Box\phi,$$

and making use of the equation of motion $\Box\phi = 0$, we can write the trace of the improved energy momentum tensor as

$$K^\mu{}_\mu = T^\mu{}_\mu + 6A\partial_\mu\phi\partial^\mu\phi = (-1+6A)\partial_\mu\phi\partial^\mu\phi = 0 \implies A = \frac{1}{6}.$$

At the end the improved energy momentum tensor reads

$$K^\mu{}_\rho = \partial_\rho\phi\partial^\mu\phi - \frac{1}{2}\delta^\mu_\rho\left(\partial_\nu\phi\partial^\nu\phi\right) + \frac{1}{6}\left(\delta^\mu_\rho\Box\phi^2 - \partial_\rho\partial^\mu\phi^2\right).$$

We can write the dilatations current S^μ in terms of the above improved energy momentum tensor

$$S^\mu = -x^\nu K^\mu_\nu - \phi\partial^\mu\phi + x^\rho\frac{1}{6}\left(\delta^\mu_\rho\Box\phi^2 - \partial_\rho\partial^\mu\phi^2\right).$$

The invariance of the theory under scale transformations implies the vanishing of $\partial_\mu S^\mu$ and therefore

$$\begin{aligned} 0 = \partial_\mu S^\mu &= -K^\mu{}_\mu - x^\nu\partial_\mu K^\mu{}_\nu - \partial_\mu\phi\partial^\mu\phi + \frac{1}{6}(4\Box\phi^2 - \Box\phi^2) \\ \implies\ K^\mu{}_\mu &= 0 \end{aligned}$$

where we have again expanded $\Box\phi^2$ and used the equation of motion $\Box\phi = 0$ and the conservation of K^ν_μ. The invariance of the theory under dilatations forces the improved energy momentum tensor to be traceless. For free theories we already know that this is the case since $K^\mu{}_\nu$ has been constructed in such a way as to have this property. However one could extend the definition of K for a more general theory with a potential

$$K^\mu{}_\rho = \partial_\rho\phi\partial^\mu\phi - \delta^\mu_\rho\left(\frac{1}{2}\partial_\nu\phi(x)\partial^\nu\phi(x) - V\right) + \frac{1}{6}\left(\delta^\mu_\rho\Box\phi^2 - \partial_\rho\partial^\mu\phi^2\right),$$

and it is possible to check that the tracelessness of K^μ_ν represents a non trivial constraint on the potential V.

The addition of a potential of the form $c_n\phi^n$ brings an additional constraint between k and d which can fix definitively the dimension. In order to have an invariant theory one needs:

$$\int d^dx'\phi'^n(x') = e^{d\lambda+nk\lambda}\int d^dx\phi^n(x) = \int d^dx\phi^n(x) \implies \begin{cases} d+nk=0 \\ k = 1-\frac{d}{2}. \end{cases}$$

The solution for the above system of equation doesn't exist for $n=2$. Instead:

$$\begin{aligned} \text{For } n=3 &\ \Rightarrow\ d=6, \\ \text{For } n=4 &\ \Rightarrow\ d=4. \end{aligned}$$

The dimensions in energy of the parameters appearing in the potential are then:

$$\begin{aligned} [\text{Action}] = E^0, &\qquad [d^dx] = E^{-d}, \qquad [\mathcal{L}] = E^d, \\ [\partial] = E, &\qquad [\phi] = E^{\frac{d}{2}-1}, \\ [m] = E, &\qquad [\beta] = E^{3-\frac{d}{2}}, \qquad [\alpha] = E^{4-d}. \end{aligned}$$

Therefore the couplings α, β are both adimensional in the dimension in which the Lagrangian is invariant under scale transformation. This is not unexpected because the scale transformation deforms lengths and energies as well. The invariance of the theory under such transformation means that the dynamics is the same at all energy scales. In order for this to be true there mustn't be any reference scale in the theory. Therefore in a scale invariant theory only dimensionless parameters are allowed in the potential. This also explains why there is no solution for the term $m^2\phi^2$: the dimension of m doesn't depend on the dimension d, hence it always introduces a reference scale which is the indeed the mass of the field.

Exercise 2

In order to classify the barions in terms of their isospin one has to decompose the tensor product of the three $j = 1/2$ representation in terms of irreducible representations of $SU(2)$. The tensor product of the bases is a basis of the tensor product space:

$$\{|\uparrow\rangle; |\downarrow\rangle\} \otimes \{|\uparrow\rangle; |\downarrow\rangle\} \otimes \{|\uparrow\rangle; |\downarrow\rangle\} = \{|\uparrow\uparrow\uparrow\rangle; |\uparrow\uparrow\downarrow\rangle; |\uparrow\downarrow\uparrow\rangle; |\downarrow\uparrow\uparrow\rangle; |\uparrow\downarrow\downarrow\rangle; |\downarrow\uparrow\downarrow\rangle; |\downarrow\downarrow\uparrow\rangle; |\downarrow\downarrow\downarrow\rangle; \}\,.$$

One can now define the representation of the $SU(2)$ generators on this space starting from the original ones:

$$\begin{aligned}
t^3 &= t^3 \otimes 1 \otimes 1 + 1 \otimes t^3 \otimes 1 + 1 \otimes 1 \otimes t^3,\\
t^+ &= t^+ \otimes 1 \otimes 1 + 1 \otimes t^+ \otimes 1 + 1 \otimes 1 \otimes t^+,\\
t^3 &= t^- \otimes 1 \otimes 1 + 1 \otimes t^- \otimes 1 + 1 \otimes 1 \otimes t^-,
\end{aligned}$$

where the original generators satisfy:

$$\begin{aligned}
&t^3|\uparrow\rangle = \frac{1}{2}|\uparrow\rangle, \qquad t^3|\downarrow\rangle = -\frac{1}{2}|\downarrow\rangle,\\
&t^+|\uparrow\rangle = 0, \qquad t^+|\downarrow\rangle = \frac{1}{\sqrt{2}}|\uparrow\rangle,\\
&t^-|\downarrow\rangle = 0, \qquad t^-|\uparrow\rangle = \frac{1}{\sqrt{2}}|\downarrow\rangle.
\end{aligned}$$

Using the the definition of t^3 one obtains that the new basis contains four different eigenvalues:

$$\begin{aligned}
\frac{3}{2} \quad &; \quad |\uparrow\uparrow\uparrow\rangle,\\
\frac{1}{2} \quad &; \quad |\uparrow\uparrow\downarrow\rangle, |\uparrow\downarrow\uparrow\rangle, |\downarrow\uparrow\uparrow\rangle,\\
-\frac{1}{2} \quad &; \quad |\uparrow\downarrow\downarrow\rangle, |\downarrow\downarrow\uparrow\rangle, |\downarrow\uparrow\downarrow\rangle,\\
-\frac{3}{2} \quad &; \quad |\downarrow\downarrow\downarrow\rangle.
\end{aligned}$$

The construction of the irreducible representation goes on as usual: first we take the highest eigenvector of t^3 and we apply the lowering operator:

$$|\uparrow\uparrow\uparrow\rangle \longrightarrow \frac{1}{\sqrt{3}}(|\downarrow\uparrow\uparrow\rangle + |\uparrow\downarrow\uparrow\rangle + |\uparrow\uparrow\downarrow\rangle) \longrightarrow \frac{1}{\sqrt{3}}(|\downarrow\downarrow\uparrow\rangle + |\downarrow\uparrow\downarrow\rangle + |\uparrow\downarrow\downarrow\rangle) \longrightarrow |\downarrow\downarrow\downarrow\rangle,$$

where we have normalized the states to 1 by hand. These states form a $j = \frac{3}{2}$ representation and in notation $|j, m\rangle$ are denoted respectively as $|\frac{3}{2}, \frac{3}{2}\rangle$, $|\frac{3}{2}, \frac{1}{2}\rangle$, $|\frac{3}{2}, -\frac{1}{2}\rangle$ and $|\frac{3}{2}, -\frac{3}{2}\rangle$. One has now to figure out how the remaining vector space decomposes under $SU(2)$. Since the space orthogonal to this representation is 4-dimensional and contains only eigenvalues $\pm 1/2$ we expect other two representation $j = 1/2$. Indeed the orthogonality conditions for the eigenvalue $+1/2$ read:

$$(\langle\downarrow\uparrow\uparrow| + \langle\uparrow\downarrow\uparrow| + \langle\uparrow\uparrow\downarrow|)\,(|\downarrow\uparrow\uparrow\rangle + A|\uparrow\downarrow\uparrow\rangle + B|\uparrow\uparrow\downarrow\rangle) = 0 \;\Rightarrow\; A = -1 - B,$$

where we have considered the general unnormalized state with $m = +1/2$. One possible choice of states, one orthogonal to the other and both orthogonal to $|\frac{3}{2}, \frac{1}{2}\rangle$, is:

$$\frac{1}{\sqrt{2}}(|\downarrow\uparrow\uparrow\rangle - |\uparrow\downarrow\uparrow\rangle)\,, \qquad \frac{1}{\sqrt{6}}(|\downarrow\uparrow\uparrow\rangle + |\uparrow\downarrow\uparrow\rangle - 2|\uparrow\uparrow\downarrow\rangle)\,.$$

These two states have $m = +1/2$ and both satisfy $t^+|\cdot\rangle = 0$, so represent the highest weight states of two distinguished $j = 1/2$ representations, so

$$\frac{1}{\sqrt{2}}(|\downarrow\uparrow\uparrow\rangle - |\uparrow\downarrow\uparrow\rangle) = \left|\frac{1}{2}, \frac{1}{2}\right\rangle^{(1)}, \qquad \frac{1}{\sqrt{6}}(|\downarrow\uparrow\uparrow\rangle + |\uparrow\downarrow\uparrow\rangle - 2|\uparrow\uparrow\downarrow\rangle) = \left|\frac{1}{2}, \frac{1}{2}\right\rangle^{(2)}.$$

Applying now the lowering operator and normalizing one gets

$$\frac{1}{\sqrt{2}}\left(|\downarrow\uparrow\downarrow\rangle - |\uparrow\downarrow\downarrow\rangle\right) = \left|\frac{1}{2},-\frac{1}{2}\right\rangle^{(1)}, \qquad \frac{1}{\sqrt{6}}\left(2|\downarrow\downarrow\uparrow\rangle - |\uparrow\downarrow\downarrow\rangle - |\downarrow\uparrow\downarrow\rangle\right) = \left|\frac{1}{2},-\frac{1}{2}\right\rangle^{(2)}.$$

All of the eight states have now been assigned to irreducible representations, so the procedure can stop. Finally, to summarize:

$$\frac{1}{2}\otimes\frac{1}{2}\otimes\frac{1}{2} = \frac{3}{2}\oplus\frac{1}{2}\oplus\frac{1}{2}.$$
$$\frac{3}{2}:\ \Delta^{++},\,\Delta^{+},\,\Delta^{0},\,\Delta^{-}, \qquad \frac{1}{2}:\ p,\,n,$$

while the second 1/2 representation is not associated to any standard particle.

Exercise 3

Consider a symmetry defined by the transformation acting on fields:

$$x' = x,$$
$$\phi'_a(x') = \mathcal{R}_a^{\ b}\phi_b(x) \simeq \phi_a(x) + i\alpha^A (T^A)_a^{\ b}\phi_b(x),$$

where $(T^A)_a^{\ b}$ are the generators of the symmetry in the appropriate representation and satisfy the Lie algebra with the ordinary commutator: $[T^A, T^B] = if^{ABC}T^C$. One can easily compute the conserved Noether's charge:

$$Q^A = \int d^3x \left(\frac{\partial\mathcal{L}}{\partial\dot{\phi}_a}\Delta_a(x)\right) = i\int d^3x\,\pi^a (T^A)_a^{\ b}\phi_b(x).$$

Therefore the Poisson brackets between two charges give:

$$\{Q^A,\, Q^B\} = \int d^3z\left(\frac{\delta Q^A}{\delta\pi^c(z)}\frac{\delta Q^B}{\delta\phi_c(z)} - \frac{\delta Q^A}{\delta\phi_c(z)}\frac{\delta Q^B}{\delta\pi^c(z)}\right).$$

Since

$$\frac{\delta Q^A}{\delta\pi(z)^c} = i\frac{\partial\left(\pi^a (T^A)_a^{\ b}\phi_b\right)}{\partial\pi^c}(z) = i(T^A)_c^{\ b}\phi_b(z),$$
$$\frac{\delta Q^B}{\delta\phi(z)_c} = i\frac{\partial\left(\pi^a (T^B)_a^{\ b}\phi_b\right)}{\partial\phi_c}(z) = i\pi^a(z)(T^B)_a^{\ c},$$

hence:

$$\{Q^A,\, Q^B\} = \int d^3z\ \pi^a\left[T^A, T^B\right]_a^{\ b}\phi_b = if^{ABC}\int d^3z\,\pi^a (T^C)_a^{\ b}\phi_b = f^{ABC}Q^C.$$

One can finally define $Q^A = -i\tilde{Q}^A$ so that

$$\{\tilde{Q}^A,\, \tilde{Q}^B\} = if^{ABC}\tilde{Q}^C.$$

There is however a shorter way to obtain the commutation rules for the charges and it involves the Jacobi identity; recall indeed that the Poisson brackets, as all the Lie products, satisfy the Jacobi relation:

$$\{\{Q^A,\, Q^B\},\phi_a\} + \{\{Q^B,\, \phi_a\}, Q^A\} + \{\{\phi_a,\, Q^A\}, Q^B\} = 0.$$

Since the charges are the generators of the transformation:

$$\{Q^A,\, \phi_a\} = \Delta_a^A = i(T^A)_a^{\ b}\phi_b,$$

then, applying two times this definition one gets

$$\{\{Q^A,\, Q^B\},\phi_a\} = -(T^B)_a^{\ c}(T^A)_c^{\ b}\phi_b + (T^A)_a^{\ c}(T^B)_c^{\ b}\phi_b = if^{ABC}(T^C)_a^{\ b}\phi_b = f^{ABC}\{Q^C,\phi_a\} = \{f^{ABC}Q^C,\phi_a\}.$$