General Relativity Formulary

1) *Differential geometry*

Metric connection (Christoffel symbols)

$$\Gamma^\alpha_{\beta\gamma} \equiv \frac{1}{2} g^{\alpha\delta} \left(\partial_\beta g_{\gamma\delta} + \partial_\gamma g_{\beta\delta} - \partial_\delta g_{\beta\gamma} \right) .$$

Covariant derivative

$$V^\alpha{}_{;\beta} = V^\alpha{}_{,\beta} + \Gamma^\alpha_{\beta\gamma} V^\gamma$$
$$V_{\alpha;\beta} = V_{\alpha,\beta} - \Gamma^\gamma_{\alpha\beta} V_\gamma .$$

Riemann curvature tensor

$$R^\alpha{}_{\beta\gamma\delta} \equiv \partial_\gamma \Gamma^\alpha_{\beta\delta} - \partial_\delta \Gamma^\alpha_{\beta\gamma} + \Gamma^\lambda_{\beta\delta} \Gamma^\alpha_{\lambda\gamma} - \Gamma^\lambda_{\beta\gamma} \Gamma^\alpha_{\delta\lambda} ,$$

or, with the first index lowered

$$\begin{aligned} R_{\alpha\beta\gamma\delta} &= g_{\alpha\lambda} R^\lambda{}_{\beta\gamma\delta} \\ &= \frac{1}{2} \left(\partial^2_{\beta\gamma} g_{\alpha\delta} + \partial^2_{\alpha\delta} g_{\beta\gamma} - \partial^2_{\alpha\gamma} g_{\beta\delta} - \partial^2_{\beta\delta} g_{\alpha\gamma} \right) \\ &\quad + g_{\mu\nu} \left(\Gamma^\mu_{\alpha\delta} \Gamma^\nu_{\beta\gamma} - \Gamma^\mu_{\alpha\gamma} \Gamma^\nu_{\beta\delta} \right) . \end{aligned}$$

Ricci tensor and Ricci scalar

$$R_{\alpha\beta} \equiv R^\gamma{}_{\alpha\gamma\beta} , \quad R \equiv g^{\alpha\beta} R_{\alpha\beta} .$$

Bianchi identities

$$R_{\alpha\beta\gamma\delta;\lambda} + R_{\alpha\beta\lambda\gamma;\delta} + R_{\alpha\beta\delta\lambda;\gamma} = 0 .$$

2) *General Relativity*

Geodesic equations

$$\frac{d^2 x^\alpha}{dp^2} + \Gamma^\alpha_{\beta\gamma} \frac{dx^\beta}{dp} \frac{dx^\gamma}{dp} = 0 .$$

Einstein equations

$$G_{\mu\nu} = 8\pi G \, T_{\mu\nu} ,$$

where $G_{\mu\nu}$ is the Einstein tensor defined by

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R \, g_{\mu\nu} ,$$

and $T_{\mu\nu}$ is the energy-momentum tensor given by

$$T_{\mu\nu} \equiv \frac{2}{\sqrt{-g}} \frac{\delta S_{\text{matter}}}{\delta g^{\mu\nu}} .$$

Variation of the determinant of the metric

$$\delta g = g \, g^{\mu\nu} \, \delta g_{\mu\nu} = -g \, g_{\mu\nu} \, \delta g^{\mu\nu} .$$

3) *Energy-momentum tensor of point-particle*

$$T^{\mu\nu} = \frac{m}{\sqrt{-g}} \int d\tau \, \frac{dz^\mu}{d\tau} \frac{dz^\nu}{d\tau} \delta^4(x - z(\tau)) .$$

4) *Schwarzschild solution*

$$ds^2 = \left(1 - \frac{2GM}{r} \right) dt^2 - \frac{1}{1 - \frac{2GM}{r}} dr^2 - r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2 \right) .$$

Geodesic equations for the Schwarzschild geometry

$$\begin{aligned} \ddot{t} + \frac{B'}{B} \dot{t} \dot{r} &= 0 \\ \ddot{r} + \frac{1}{2} \frac{B'}{A} \dot{t}^2 + \frac{1}{2} \frac{A'}{A} \dot{r}^2 - \frac{r}{A} \dot{\theta}^2 - \frac{r \sin^2\theta}{A} \dot{\phi}^2 &= 0 \\ \ddot{\theta} + \frac{2}{r} \dot{r} \dot{\theta} - \sin\theta \cos\theta \dot{\phi}^2 &= 0 \\ \ddot{\phi} + \frac{2}{r} \dot{r} \dot{\phi} + 2 \cot\theta \, \dot{\theta} \dot{\phi} &= 0, \end{aligned}$$

where

$$B(r) = 1 - \frac{2GM}{r}, \quad A(r) = B^{-1}(r).$$

Motion in the Schwarzschild metric

$$A(r) \left(\frac{dr}{d\phi} \right)^2 \frac{J^2}{r^4} + \frac{J^2}{r^2} - \frac{1}{B(r)} = -E$$

$$d\tau^2 = E B^2(r) dt^2, \quad r^2 \frac{d\phi}{dt} = J B(r)$$

Massive particle

$$\frac{1}{2} \left(\frac{dr}{d\tau} \right)^2 + \frac{1}{2} \left(1 - \frac{2GM}{r} \right) \left(1 + \frac{J^2/E}{r^2} \right) = \frac{1}{2E}$$

Massless particle ($d\lambda = B(r) dt$)

$$\frac{1}{2} \left(\frac{dr}{d\lambda} \right)^2 + \frac{1}{2} \left(1 - \frac{2GM}{r} \right) \frac{J^2}{r^2} = \frac{1}{2}$$

5) *Linearized theory*

$$\begin{aligned} g_{\mu\nu} &= \eta_{\mu\nu} + h_{\mu\nu} \\ R^{(1)}_{\mu\nu} &= \frac{1}{2} \left(-\Box h_{\mu\nu} + \partial_\lambda \partial_\mu h^\lambda{}_\nu + \partial_\lambda \partial_\nu h^\lambda{}_\mu - \partial_\mu \partial_\nu h \right) \\ R^{(1)} &= -\Box h + \partial_\mu \partial_\nu h^{\mu\nu} \end{aligned}$$

Einstein equations for physical amplitudes

$$\begin{aligned} \Box h_{12} &= 8\pi G \, T_{12} \\ \Box h &= 4\pi G (T_{22} - T_{11}) \end{aligned}$$

6) *Numerical values*

$$\begin{aligned} G &= 6.67 \cdot 10^{-11} \, \mathrm{m^3 \, kg^{-1} \, s^{-2}} \\ c &\simeq 3 \cdot 10^8 \, \mathrm{m/s} \\ M_\odot &\simeq 2 \cdot 10^{30} \, \mathrm{kg}, \; r_\odot \simeq 7 \cdot 10^5 \, \mathrm{km} \\ M_\oplus &\simeq 6 \cdot 10^{24} \, \mathrm{kg}, \; r_\oplus \simeq 6000 \, \mathrm{km} \\ M_{NS} &\simeq 2 M_\odot, \; r_{NS} \simeq 12 \, \mathrm{km} \end{aligned}$$