General Relativity Formulary

1) *Differential geometry*
Metric connection (Christoffel symbols)

$$\Gamma^{\alpha}_{\beta\gamma} \equiv \frac{1}{2} g^{\alpha\delta} \left(\partial_{\beta} g_{\gamma\delta} + \partial_{\gamma} g_{\beta\delta} - \partial_{\delta} g_{\beta\gamma} \right) .$$

Covariant derivative

$$V^{\alpha}{}_{;\beta} = V^{\alpha}{}_{,\beta} + \Gamma^{\alpha}_{\beta\gamma} V^{\gamma}$$
$$V_{\alpha;\beta} = V_{\alpha,\beta} - \Gamma^{\gamma}_{\alpha\beta} V_{\gamma} .$$

Riemann curvature tensor

$$R^{\alpha}{}_{\beta\gamma\delta} \equiv \partial_{\delta}\Gamma^{\alpha}_{\beta\gamma} - \partial_{\gamma}\Gamma^{\alpha}_{\beta\delta} + \Gamma^{\lambda}_{\beta\gamma}\Gamma^{\alpha}_{\delta\lambda} - \Gamma^{\lambda}_{\beta\delta}\Gamma^{\alpha}_{\lambda\gamma} ,$$

or, with the first index lowered

$$\begin{aligned} R_{\alpha\beta\gamma\delta} &= g_{\alpha\lambda} R^{\lambda}{}_{\beta\gamma\delta} \\ &= \frac{1}{2}\left(\partial^2_{\alpha\gamma} g_{\beta\delta} + \partial^2_{\beta\delta} g_{\alpha\gamma} - \partial^2_{\beta\gamma} g_{\alpha\delta} - \partial^2_{\alpha\delta} g_{\beta\gamma} \right) \\ &\quad + g_{\mu\nu}\left(\Gamma^{\mu}_{\alpha\gamma}\Gamma^{\nu}_{\beta\delta} - \Gamma^{\mu}_{\alpha\delta}\Gamma^{\nu}_{\beta\gamma} \right) . \end{aligned}$$

Ricci tensor and Ricci scalar

$$R_{\alpha\beta} \equiv R^{\gamma}{}_{\alpha\gamma\beta} , \quad R \equiv g^{\alpha\beta} R_{\alpha\beta} .$$

Bianchi identities

$$R_{\alpha\beta\gamma\delta;\lambda} + R_{\alpha\beta\lambda\gamma;\delta} + R_{\alpha\beta\delta\lambda;\gamma} = 0 .$$

2) *General Relativity*
Geodesic equations

$$\frac{d^2 x^{\alpha}}{dp^2} + \Gamma^{\alpha}_{\beta\gamma} \frac{dx^{\beta}}{dp}\frac{dx^{\gamma}}{dp} = 0 ,$$

where $p = \tau$ for a massive particle and $p = x^0$ for a massless particle.

$T_{\mu\nu}$ is the energy-momentum tensor given by

$$T_{\mu\nu} \equiv \frac{2}{\sqrt{-g}} \frac{\delta S_{\text{matter}}}{\delta g^{\mu\nu}} .$$

Variation of the determinant of the metric

$$\delta g = g\, g^{\mu\nu}\, \delta g_{\mu\nu} = -g\, g_{\mu\nu}\, \delta g^{\mu\nu} .$$

3) *Energy-momentum tensor of point-particle*

$$T^{\mu\nu} = \frac{m}{\sqrt{-g}} \int d\tau \frac{dz^{\mu}}{d\tau}\frac{dz^{\nu}}{d\tau} \delta^4(x - z(\tau)) .$$

4) *Static isotropic spacetime*

$$ds^2 = B(r)dt^2 - A(r)dr^2 - r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2 \right) .$$

The corresponding Geodesic equations

$$\begin{aligned} \ddot{t} + \frac{B'}{B}\dot{t}\dot{r} &= 0 \\ \ddot{r} + \frac{1}{2}\frac{B'}{A}\dot{t}^2 + \frac{1}{2}\frac{A'}{A}\dot{r}^2 - \frac{r}{A}\dot{\theta}^2 - \frac{r\sin^2\theta}{A}\dot{\phi}^2 &= 0 \\ \ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} - \sin\theta\cos\theta\dot{\phi}^2 &= 0 \\ \ddot{\phi} + \frac{2}{r}\dot{r}\dot{\phi} + 2\cot\theta\,\dot{\theta}\dot{\phi} &= 0, \end{aligned}$$

4.1) *Schwarzschild solution*

$$B(r) = 1 - \frac{2GM}{r}, \quad A(r) = B^{-1}(r).$$

Motion in Schwarzschild solution

$$A(r)\left(\frac{dr}{d\phi}\right)^2 \frac{J^2}{E^2 r^4} + \frac{J^2}{E^2 r^2} - \frac{1}{B(r)} = -\frac{m^2}{E^2}$$
$$d\tau^2 = \frac{m^2 B^2(r)}{E^2} dt^2, \quad r^2 \frac{d\phi}{dt} = \frac{J}{E} B(r)$$

Radial equation for a massive particle

$$\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + \frac{1}{2}\left(1 - \frac{2GM}{r}\right)\left(1 + \frac{J^2/m^2}{r^2}\right) = \frac{E^2}{2m^2}$$

Radial equation for a massless particle ($d\lambda = B(r)dt$)

$$\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + \frac{1}{2}\left(1 - \frac{2GM}{r}\right)\frac{J^2}{E^2 r^2} = \frac{1}{2}$$

5) *Numerical values*

$$G = 6.67 \cdot 10^{-11}\, \text{m}^3\, \text{kg}^{-1}\, \text{s}^{-2}$$
$$M_{\odot} \simeq 2 \cdot 10^{30}\, \text{kg}, \; r_{\odot} \simeq 7 \cdot 10^5\, \text{km}$$
$$M_{\oplus} \simeq 6 \cdot 10^{24}\, \text{kg}, \; r_{\oplus} \simeq 6000\, \text{km}$$
$$M_{NS} \simeq 2M_{\odot}, \; r_{NS} \simeq 12\, \text{km}$$