Exercice 1 *Image charge method - I*

i) We need three image charges; due to the symmetry of the problem, we can put them in the positions $(-x_0, y_0)$, $(x_0, -y_0)$, $(-x_0, -y_0)$. The Green's function of the straight line wire in absence of the conducting planes is

$$G(\mathbf{x}, \mathbf{x}') = \log \frac{R^2}{r^2},$$

where R is a constant and $r^2 = (x^2 - x'^2) + (y - y')^2$. The Green's function for the problem with the three image charges is therefore easily found to be:

$$\begin{aligned} G(\mathbf{x}, \mathbf{x}') = \quad & \log\left(\frac{R^2}{(x-x')^2 + (y-y')^2}\right) - \log\left(\frac{R^2}{(x+x')^2 + (y-y')^2}\right) \\ & - \log\left(\frac{R^2}{(x-x')^2 + (y+y')^2}\right) + \log\left(\frac{R^2}{(x+x')^2 + (y+y')^2}\right), \end{aligned}$$

which satisfies the Dirichlet boundary conditions $G(x = 0, y) = 0$ and $G(x, y = 0) = 0$. The potential is therefore:

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int d^2x' \lambda \delta(x' - x_0)\delta(y' - y_0) G(x, x') = \frac{\lambda}{4\pi\epsilon_0} G(\mathbf{x}, \mathbf{x_0}).$$

ii)The charge density induced on the plane $(x \geq 0, y = 0)$ can be related to the electric field which is normal to this surface:

$$\sigma(\mathbf{x}) = \epsilon_0 E_y(\mathbf{x})|_{y=0}.$$

From the expression of the potential, we easily find:

$$\sigma(x) = \epsilon_0 \left(-\frac{\partial \Phi(\mathbf{x})}{\partial y}\right)|_{y=0} = -\frac{\lambda y_0}{\pi}\left[\frac{1}{(x-x_0)^2 + y_0^2} - \frac{1}{(x+x_0)^2 + y_0^2}\right].$$

Finally, the total charge is obtained integrating the chage density over the whole plane:

$$Q = \int_0^\infty dx \sigma(x) = -\frac{2}{\pi}\lambda \arctan\left(\frac{x_0}{y_0}\right).$$

Due to the symmetry between the x and y axis, the total charge on the other conducting plane must be:

$$Q = -\frac{2}{\pi}\lambda \arctan\left(\frac{y_0}{x_0}\right).$$

iii)For solving this question, it is useful to rewrite the potential in cylindrical coordinates, (ρ, θ, z). We find:

$$\begin{aligned} \Phi(\rho, \theta; \rho_0, \theta_0) \quad &= -\frac{\lambda}{4\pi\epsilon_0}\left[\log\left(1 + \frac{\rho_0^2}{\rho^2} - 2\frac{\rho_0}{\rho}\cos(\theta - \theta_0)\right) - \log\left(1 + \frac{\rho_0^2}{\rho^2} + 2\frac{\rho_0}{\rho}\cos(\theta + \theta_0)\right)\right. \\ & \left. - \log\left(1 + \frac{\rho_0^2}{\rho^2} - 2\frac{\rho_0}{\rho}\cos(\theta + \theta_0)\right) + \log\left(1 + \frac{\rho_0^2}{\rho^2} + 2\frac{\rho_0}{\rho}\cos(\theta - \theta_0)\right)\right]. \end{aligned}$$

Expanding in a Taylor series for $\rho \gg \rho_0$, we can approximate the potential as:

$$\Phi = -\frac{\lambda}{\pi\epsilon_0}\left(\frac{\rho_0}{\rho}\right)^2 (-4\cos\theta\cos\theta_0\sin\theta\sin\theta_0) = \frac{4\lambda}{\pi\epsilon_0}(x_0 y_0)\frac{xy}{\rho^4}.$$

This is a quadrupole term, which is generated due to the presence of four line charges (the real one plus three imaginary ones).

Exercice 2 *Forces on a conductor*

i) We can write

$$\begin{aligned}\rho E_i &= \epsilon_0(\nabla\cdot\mathbf{E})E_i = \epsilon_0(\nabla^2\Phi)\nabla_i\Phi = \epsilon_0\nabla_j\left[(\nabla_j\Phi)\nabla_i\Phi\right] - \epsilon_0\left[(\nabla_j\Phi)\nabla_j\nabla_i\Phi\right] \quad (1)\\ &= \epsilon_0\nabla_j\left[(\nabla_j\Phi)\nabla_i\Phi\right] - \frac{\epsilon_0}{2}\nabla_i\left[(\nabla\Phi)^2\right] = \text{ total derivative} \quad (2)\end{aligned}$$

ii) Enclosing the charge density in some finite volume V we can write

$$\begin{aligned}\int_V d^3x\rho E_i &= \epsilon_0\int_V d^3x\left(\nabla_j\left[(\nabla_j\Phi)\nabla_i\Phi\right] - \frac{1}{2}\nabla_i\left[(\nabla\Phi)^2\right]\right) \quad (3)\\ &= \epsilon_0\int_{\partial V}(\nabla_i\Phi)\left(\nabla\Phi\cdot d\mathbf{a}\right) - \frac{\epsilon_0}{2}\int_{\partial V}(\nabla\Phi)^2 d\mathbf{a}_i \quad (4)\end{aligned}$$

where in the last equality we have used the divergence theorem.

iii) Let's now we restrict ourselves to a finite conductor. In calculating the force we can choose to integrate over any volume (so long at it contains the entire surface of the conductor). A natural choice is to choose a volume that corresponds to that of the conductor (V) plus a small infinitesimal additional piece to avoid dealing with possible ambiguities on the boundary (ϵ). Let's denote the total volume V_+.

Now, as the electric field on the surface of a conductor is perpendicular to the surface when utilize the identity derived in ii) and take the $\epsilon -> 0$ we see that both terms yield the same result. Therefore, we have that

$$\text{force on the conductor} = \int_V d^3x\rho E_i = \frac{\epsilon_0}{2}\int_{\partial V_+}(\nabla\Phi)^2 d\mathbf{a}_i = \frac{\epsilon_0}{2}\int_{\partial V_+}(\mathbf{E})^2 d\mathbf{a}_i\ . \quad (5)$$

Notice that the ambiguity created by the singular nature of the surface has been resolved.

Exercice 3 *Image charge method - II*

i) We need two image charges, one for each one of the real charges producing the constant external electric field. Following the solution of the electrostatic problem of a point charge and a spherical conductor discussed during the course, we need to place a positive image charge with charge Qa/R in the position $z = -a^2/R$ and a negative image charge with charge $-Qa/R$ in the position $z = a^2/R$. The potential outside the spherical uncharged conductor in a constant electric field is therefore the potential produced by all the four charges; in spherical coordinates, we have:

$$\begin{aligned}\Phi(r,\theta) = \ &\frac{Q}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{r^2+R^2+2rR\cos\theta}} - \frac{1}{\sqrt{r^2+R^2-2rR\cos\theta}}\right.\\ &\left.+\frac{a/R}{\sqrt{r^2+\frac{a^4}{R^2}-2a^2\frac{r}{R}\cos\theta}} - \frac{a/R}{\sqrt{r^2+\frac{a^4}{R^2}+2a^2\frac{r}{R}\cos\theta}}\right).\end{aligned}$$

It is immediate to verify that for $r = a$ the potential becomes zero, as required by imposing the Dirichlet boundary condition. In the limit $R \gg r$, which reproduces the constant electric field, we can Taylor expand the above expression and find:

$$\Phi(r,\theta) = -E_0\left(r - \frac{a^3}{r^2}\right)\cos\theta,$$

where we usefully defined $E_0 = 2Q/(4\pi\epsilon_0 R^2)$. The first term is just the potential due to the external field in spherical coordinates, the second term is the potential of a perfect dipole produced by the induced charged distribution on the sphere.

ii) If the sphere is now cut into two hemispheres, the bottom hemisphere will feel a total force:

$$\mathbf{F} = \int da\sigma(\mathbf{x})\mathbf{E}(\mathbf{x}),$$

where σ is the charge density induced on the sphere by the external electric field and $\mathbf{E}$ is the electric field on the hemisphere. In order not to take into account the force of the bottom hemisphere on itself, we can express the electric field on the surface of the conductor due to non-self contributions as $\mathbf{E} = \frac{\sigma}{2\epsilon_0}\widehat{\mathbf{e}}_\mathbf{r}$. So, we have:

$$\mathbf{F} = \frac{1}{2\epsilon_0}\int da\sigma(\mathbf{x})^2\widehat{\mathbf{e}}_\mathbf{r}.$$

Since

$$\mathbf{E}(\mathbf{x}) = -\nabla\Phi(r,\theta) = E_0\left[\left(1 + 2\frac{a^3}{r^3}\right)\cos\theta\widehat{\mathbf{e}}_\mathbf{r} + \left(\frac{a^3}{r^3} - 1\right)\sin\theta\widehat{\mathbf{e}}_{\boldsymbol{\theta}}\right],$$

the electric field on the surface of the sphere is

$$\mathbf{E}(r = a,\theta) = 3E_0\cos\theta\widehat{\mathbf{e}}_\mathbf{r},$$

which induces the charge density $\sigma = 3\epsilon_0 E_0\cos\theta$. The force needed to hold the bottom hemisphere is therefore:

$$\mathbf{F} = \frac{a^2}{2\epsilon_0}\widehat{\mathbf{e}}_\mathbf{z}\int_{\pi/2}^{\pi}\int_0^{2\pi}(3\epsilon_0 E_0\cos\theta)^2\cos\theta\sin\theta d\theta d\phi.$$

In the last step, we have projected to the z component of F which is the only non-zero component of the force, due to the spherical symmetry of the problem. Carrying out the integration, we easily find:

$$\mathbf{F} = -\frac{9}{4}\pi\epsilon_0 E_0^2 a^2\widehat{\mathbf{e}}_\mathbf{z}.$$

So, the force needed to keep the bottom hemisphere in position is must therefore be oriented in the positive z direction with the same modulus of $\mathbf{F}$. Due to the symmetry of the problem, the force needed to hold the other hemisphere in position must be equal and opposite.

iii)If the sphere has an total charge Q, it will spread uniformly on the sphere as an additional charge to the induced one:

$$\sigma_0 = \frac{Q}{4\pi a^2}.$$

The additional force on the hemisphere will therefore be:

$$\mathbf{F_2} = \frac{a^2}{2\epsilon_0}\widehat{\mathbf{e}}_\mathbf{z}\int_{\pi/2}^{\pi}\int_0^{2\pi}\sigma_0^2\cos\theta\sin\theta d\theta d\phi = \frac{Q^2}{32\pi\epsilon_0 a^2}.$$

The total force needed to keep the two hemispheres together is the sum of the two contributions:

$$\mathbf{F_{total}} = \widehat{\mathbf{e}}_\mathbf{z}\left(\frac{9}{4}\pi a^2\epsilon_0 E_0^2 + \frac{Q^2}{32\pi\epsilon_0 a^2}\right).$$

Exercice 4 *Lorentz transformation*

i) In the rest frame K' there's no current and the electric field is static; hence $\mathbf{B'} = 0$. The electric field is found using Gauss's theorem and is given by:

$$\mathbf{E'} = \frac{\lambda}{2\pi\epsilon_0 r}\widehat{\mathbf{e}}_\mathbf{r},$$

where r is distance of the observer from the wire in cylindrical coordinates. In order to find the components of the electric and magnetic fields in the laboratory frame k, we perform a boost along the z axis with $\boldsymbol{\beta} = (v/c)\widehat{\mathbf{e}}_{\mathbf{z}}$. The fields in K are related to those in K' by the usual transformations:

$$\mathbf{E} = \gamma(\mathbf{E}' - \boldsymbol{\beta} \times \mathbf{B}') - \frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta} \cdot \mathbf{E}'),$$
$$\mathbf{B} = \gamma(\mathbf{B}' + \boldsymbol{\beta} \times \mathbf{E}') - \frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta} \cdot \mathbf{B}').$$

Using the above expressions of the electric and magnetic fields in the rest frame, we easily find:

$$\mathbf{E} = \gamma\mathbf{E}' = \gamma\frac{\lambda}{2\pi\epsilon_0 r}\widehat{\mathbf{e}}_{\mathbf{r}}, \qquad \mathbf{B} = \gamma\boldsymbol{\beta} \times \mathbf{E}' = \gamma\beta\frac{\lambda}{2\pi\epsilon_0 r}\widehat{\mathbf{e}}_{\phi}.$$

ii)In the rest frame, the charge density is $\rho' = \lambda\delta(x)\delta(y)$ and there's no current, so that

$$J'^{\mu} = c\lambda(\delta(x)\delta(y), 0, 0, 0).$$

Performing a boost in the laboratory frame, we find:

$$J^{\mu} = \Lambda^{\mu}{}_{\nu} J'^{\nu} = c\lambda \begin{pmatrix} \gamma & 0 & 0 & \gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta\gamma & 0 & 0 & \gamma \end{pmatrix} \cdot \begin{pmatrix} \delta(x)\delta(y) \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \delta(x)\delta(y) \\ 0 \\ 0 \\ \beta\delta(x)\delta(y) \end{pmatrix}.$$

iii)Computing the electric field in the laboratory frame is easy, since the charge density is just the original one re-scaled by factor of γ. The electric field is therefore the same we derived through an explicit Lorentz boost. The magnetic field is produced by the charge density that appears in the laboratory frame, $\mathbf{J} = c\beta\gamma\lambda\delta(x)\delta(y)\widehat{\mathbf{e}}_{\mathbf{z}}$. Using Ampère's law, we find

$$\mathbf{B} = \gamma\beta\frac{\lambda}{2\pi\epsilon_0 r}\widehat{\mathbf{e}}_{\phi},$$

which again agrees with the result previously found.

Exercice 5 *Symmetry*

i) We can view the solenoid as stack of identical (not necessarily cylindrical) loops. In order to determine the magnetic field at some point $\mathbf{r}$ we need to integrate over this infinite stack of loops each carrying and infinitesimal amount of current. That is, we can the total magnetic field can be represented by

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_{length} n\,dz \int_{loop} I\frac{d\mathbf{l}' \times \mathbf{x}}{|x|^3} \tag{6}$$

where $\mathbf{x}$ is the distance from the source (on the solenoid) to the point $\mathbf{r}$.

For simplicity let's choose our coordinates such that $\mathbf{r}$ lies on the y axis, say at the point $(0, y, 0)$, with the solenoid oriented in the z direction. Let's look at the contribution coming from one loop at some height z', in particular, consider the contribution of just one segment of that loop at the point (x', y', z'): $\mathbf{x} = (-x', y - y', -z')$ and $d\mathbf{l}' = (dx', dy', 0)$ and so

$$d\mathbf{l}' \times \mathbf{x} = (-z'dy', z'dx', (y - y')dx' + x'dy') \tag{7}$$

$$\Longrightarrow \qquad d\mathbf{B}(\mathbf{r})_{loop} = \frac{\mu_0 I}{4\pi}\frac{(-z'dy', z'dx', (y - y')dx' + x'dy')}{|x|^3} \tag{8}$$

Now consider a symmetrically placed source element on an identical loop which is instead at $-z'$, its position is (x', y', z'). Looking at the above expression we see that for this "mirror image" loop $d\mathbf{B}$ is identical with the signs flipped in the x and y components. And so, when we add the contribution of these two sources together we are left with *only* the z component.

That is, for every segment of wire there is a symmetrical piece (at $-z$) whose contribution exactly cancels the component of the magnetic field in the plane perpendicular to the axis. When we integrate over all the loops we are left with a magnetic field that points only parallel to the axis of the solenoid.

ii) Now that we know that $\mathbf{B}$ points only in the axis of the solenoid we can apply Ampere's law with ease. We get exactly the same result as we would for a cylindrical solenoid:

$$\mathbf{B} = \begin{cases} \mu_0 n I \hat{z} & \text{inside the solenoid} \\ 0 & \text{outside the solenoid} \end{cases} \tag{9}$$

Exercice 6 *Multipoles*

As the name suggests this is an exercise in the multipole expansion. If one has a localized source of some typical length scale w and one is at a very large typical distance $r \gg w$ it makes sense to expand

$$\int d^3x' \frac{source(x')}{|x' - x|} \tag{10}$$

in a power series of terms that scale as increasing powers of (w/r). This is the multipole expansion.

In our particular case, we have that

$$\begin{aligned} \mathbf{A}(\mathbf{r}) &= \frac{\mu_0}{4\pi} \int d^3x' \frac{\mathbf{J}(x')}{|x' - x|} &(11)\\ &= \frac{\mu_0}{4\pi} \left(\frac{r_i}{r^3} \mathbf{d}_i + \frac{r_i r_j}{r^5} \mathbf{Q}_{ij} + ... \right) &(12) \end{aligned}$$

where $\mathbf{d}_i = \int d^3x' \mathbf{J}(x') r'_i$ and $\mathbf{Q}_{ij} = \int d^3x' \mathbf{J}(x') \frac{1}{2} \left(3r'_i r'_j - r'^2 \delta_{ij} \right)$ — sort of vector dipole and quadrupole moments. We have dropped the monopole term. Now, if we are considering a current density along a single wire, the expressions become even more simple as $d^3x' \mathbf{J}(x') \to I \int d\mathbf{l}$. Using Greens identity ($\int_S \nabla T \times d\mathbf{a} = -\int_P T d\mathbf{l}$) the first term can be written as

$$\frac{\mu_0}{4\pi} \frac{m \times r}{r^3} \tag{13}$$

where $\mathbf{m}$ is the usual magnetic dipole moment $\mathbf{m} = I\mathbf{a}$.

For our particular configuration at hand, it is easiest to envision it as the sum of two squares (with sides of length w) adjoined at a 90 degree angle.

i) From this perspective the leading order behavior at large distances is given by the dipole moment which is given simply by the dipole moment of one square added to the other (why do we not have to worry about which point the dipole moment is calculated about?). That is, in the coordinates chosen in the figure,

$$\mathbf{m} = I w^2 (\hat{y} + \hat{z}) \tag{14}$$

and so we can simply take the curl of the vector potential to obtain the magnetic field:

$$\mathbf{B}_{dip}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{1}{r^3} \left[3(\mathbf{m} \cdot \hat{\mathbf{r}}) \hat{\mathbf{r}} - \mathbf{m} \right] . \tag{15}$$

ii) The leading order correction is given by the quadrupole term. In order to be more explicit, we will put indices also on the $\mathbf{J}$, that is, let's define

$$Q_{n;ij} \equiv I \int \frac{1}{2} \left(3r'_i r'_j - r'^2 \delta_{ij} \right) dl_n . \tag{16}$$

We can view this guy as simply 3 traceless 3x3 matrices. Let's calculate them...

Notice first that for $Q_{x;ij}$ we only need to integrate over 2 sides (the two sides parallel to the x-axis). Similarly for $Q_{y;ij}$ and $Q_{z;ij}$.

When we attempt to compute these quantities we realize that all diagonal terms simply vanish. Continuing we find that:

$$\begin{aligned}
Q_{x;ij} &= 0 && (17)\\
Q_{y;ij} &= Iw^3 \begin{pmatrix} 0 & \frac{3}{2} & 0 \\ \frac{3}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} && (18)\\
Q_{z;ij} &= Iw^3 \begin{pmatrix} 0 & 0 & -\frac{3}{2} \\ 0 & 0 & 0 \\ -\frac{3}{2} & 0 & 0 \end{pmatrix} && (19)
\end{aligned}$$

Plugging this in, we find the leading correction to the vector potential to be

$$\begin{aligned}
\delta A_x &= 0 && (20)\\
\delta A_y &= \frac{3\mu_0 I}{4\pi}\frac{w^3}{r^5}xy && (21)\\
\delta A_z &= -\frac{3\mu_0 I}{4\pi}\frac{w^3}{r^5}xz && (22)
\end{aligned}$$

One can take the curl of this to obtain the correction in the magnetic field at long distances. Notice that indeed $\delta\mathbf{A}$ is down by a factor of (w/r) compared to the leading vector potential.