1 Exercice 1

1.1

$$\int_S d\omega_1 = \int_{\partial S} \omega_1, \qquad \omega_1 = v_i dy^i$$

- $$\int_S d\omega_1 = \int_S \frac{\partial v_j}{\partial y^i}\, dy^i \wedge dy^j = \int_S \frac{\partial v_j}{\partial y^i}\frac{\partial y^i}{\partial x^a}\frac{\partial y^j}{\partial x^b}\, dx^a \wedge dx^b =$$
$$= \int_S \partial_i v_j \frac{\partial y^i}{\partial x^a}\frac{\partial y^j}{\partial x^b}\,\epsilon^{ab}\, dx^1 \wedge dx^2 = \int_S \partial_i v_j \left(\frac{\partial y^i}{\partial x^1}\frac{\partial y^j}{\partial x^2} - \frac{\partial y^i}{\partial x^2}\frac{\partial y^j}{\partial x^1}\right) dx^1 \wedge dx^2 =$$
$$= \int_S \partial_i v_j \left(\delta^i_m \delta^j_n - \delta^i_n \delta^j_m\right)\frac{\partial y^m}{\partial x^1}\frac{\partial y^n}{\partial x^2}\, dx^1 \wedge dx^2 = \int_S \partial_i v_j\, \epsilon^{ijk}\epsilon_{mnk}\frac{\partial y^m}{\partial x^1}\frac{\partial y^n}{\partial x^2}\, dx^1 \wedge dx^2 =$$
$$= \int_S \left(\vec{\nabla} \times \vec{v}\right) \cdot \left(\frac{\partial \vec{y}}{\partial x^1} \times \frac{\partial \vec{y}}{\partial x^2}\right) dx^1 \wedge dx^2 = \int_S \left(\vec{\nabla} \times \vec{v}\right) \cdot \left(\frac{\partial \vec{y}}{\partial x^1} \times \frac{\partial \vec{y}}{\partial x^2}\right)(\sqrt{|g|})^{-1}\Omega_2$$

Où on a multiplié et divisé par $\sqrt{|g|}$ où g est le déterminant de la métrique induite sur la surface S. En utilisant la définition d'intégration des formes on a:

$$\int_S d\omega_1 = \int_S \left(\vec{\nabla} \times \vec{v}\right) \cdot \left(\frac{\partial \vec{y}}{\partial x^1} \times \frac{\partial \vec{y}}{\partial x^2}\right) dx^1 dx^2 = \oint_S \left(\vec{\nabla} \times \vec{v}\right) \cdot d\vec{\sigma}$$

- $$\int_{\partial S} \omega_1 = \int_{\partial S} v_i\, dy^i = \int_{\partial S} v_i \frac{\partial y^i}{\partial x^a}\, dx^a = \int_{\partial S} \vec{v} \cdot d\vec{s}$$

1.2

$$\int_V d * \omega_1 = \int_{\partial V} *\omega_1, \qquad \omega_1 = v_i dy^i$$

$$*\omega_1 = \frac{1}{2} v^i \epsilon_{ijk}\, dy^j \wedge dy^k, \qquad d * \omega_1 = \frac{1}{2}\frac{\partial v^i}{\partial y^l}\epsilon_{ijk}\, dy^l \wedge dy^j \wedge dy^k$$

En utilisant le fait que:

$$dy^l \wedge dy^j \wedge dy^k = \epsilon^{ljk} dy^1 \wedge dy^2 \wedge dy^3$$

on trouve:

- $$\int_V d * \omega_1 = \int_V \frac{1}{2}\frac{\partial v^i}{\partial y^l}\epsilon_{ijk}\epsilon^{ljk}\, dy^1 \wedge dy^2 \wedge dy^3 = \int_V \vec{\nabla} \cdot \vec{v}\, dy^1 \wedge dy^2 \wedge dy^3 \equiv \int_V \vec{\nabla} \cdot \vec{v}\, dV$$
- $$\int_{\partial V} *\omega_1 = \int_{\partial V} \frac{1}{2} v^i \epsilon_{ijk}\, dy^j \wedge dy^k = \int_{\partial V} \frac{1}{2} v^i \epsilon_{ijk} \left(\frac{\partial y^j}{\partial x^1}\frac{\partial y^k}{\partial x^2} - \frac{\partial y^j}{\partial x^2}\frac{\partial y^k}{\partial x^1}\right) dx^1 \wedge dx^2 =$$
$$= \int_{\partial V} \frac{1}{2} v^i\, \epsilon_{ijk}\, \epsilon^{jkl}\, \epsilon_{mnl}\frac{\partial y^m}{\partial x^1}\frac{\partial y^n}{\partial x^2}\, dx^1 \wedge dx^2 = \int_{\partial V} v^l \left(\frac{\partial \vec{y}}{\partial x^1} \times \frac{\partial \vec{y}}{\partial x^2}\right)_l dx^1 \wedge dx^2$$
$$= \int_{\partial V} \vec{v} \cdot \left(\frac{\partial \vec{y}}{\partial x^1} \times \frac{\partial \vec{y}}{\partial x^2}\right) dx^1 \wedge dx^2 \equiv \int_{\partial V} \vec{v} \cdot \left(\frac{\partial \vec{y}}{\partial x^1} \times \frac{\partial \vec{y}}{\partial x^2}\right) dx^1 dx^2 = \int_{\partial V} \vec{v} \cdot d\vec{\sigma}$$

2 Exercice 2

$$F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu$$

$$d * F = \frac{1}{4} \partial_\sigma F^{\mu\nu}\, \epsilon_{\mu\nu\alpha\beta}\, dx^\sigma \wedge dx^\alpha \wedge dx^\beta$$

$$*d * F = \frac{1}{4} \partial_\sigma F^{\mu\nu}\, \epsilon_{\mu\nu\alpha\beta}\, \epsilon^{\sigma\alpha\beta}{}_{\rho}\, dx^\rho$$

$$\epsilon_{\mu\nu\alpha\beta}\, \epsilon^{\sigma\alpha\beta}{}_{\rho}\, dx^\rho = \epsilon_{\mu\nu\alpha\beta}\, \epsilon^{\sigma\alpha\beta\tau}\, \eta_{\tau\rho} dx^\rho = -2\big[\delta^\tau_\mu \delta^\sigma_\nu - \delta^\tau_\nu \delta^\sigma_\mu\big] dx^\rho \eta_{\tau\rho} = -2\big[\eta_{\mu\rho}\delta^\sigma_\nu - \eta_{\nu\rho}\delta^\sigma_\mu\big] dx^\rho$$

$$*d * F = -\frac{1}{2} \partial_\sigma F^{\mu\nu} \left[\eta_{\mu\rho}\delta^\sigma_\nu - \eta_{\nu\rho}\delta^\sigma_\mu\right] dx^\rho = -\frac{1}{2}\big(\partial_\sigma F_\rho{}^\sigma dx^\rho - \partial_\sigma F^\sigma{}_\rho dx^\rho\big)$$

$$J = J_\rho dx^\rho$$

- $$* \, d * F = \frac{4\pi}{c} J \quad \Rightarrow \quad \partial_\sigma F^{\sigma\rho} = \frac{4\pi}{c} J^\rho$$
- $$dF = 0 \quad \Rightarrow \quad dF \wedge dx^\sigma = 0 \quad \Rightarrow \quad \frac{1}{2} \partial_\alpha F_{\mu\nu}\, dx^\alpha \wedge dx^\mu \wedge dx^\nu \wedge dx^\sigma = 0$$
  $$\Rightarrow \quad \frac{1}{2} \partial_\alpha F_{\mu\nu}\, \epsilon^{\alpha\mu\nu\sigma}\, dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 = 0 \quad \Rightarrow \quad \epsilon^{\sigma\alpha\mu\nu}\, \partial_\alpha F_{\mu\nu} = 0$$