1 Exercice 1

1.1

Pour montrer que la loi de transformation correcte est:

$$\mathcal{O}'^{\,i} = \big(R(\boldsymbol{\alpha})^{-1}\big)^{i}_{\ j}\ \mathcal{O}^{j}$$

on peut appliquer une deuxième transformation:

$$\mathcal{O}''^{\,i} \equiv \phi_d\big(R(\boldsymbol{\beta})\big)\,\mathcal{O}'^{i}\,\phi_d\big(R(\boldsymbol{\beta})\big)^{-1}$$

Il faut alors vérifier que:

$$\mathcal{O}''^{\,i} = \Big(\big(R(\boldsymbol{\beta})\circ R(\boldsymbol{\alpha})\big)^{-1}\Big)^{i}_{\ j}\ \mathcal{O}^{j}$$

Donc:

$$\begin{aligned}\mathcal{O}''^{\,i} \equiv \phi_d\big(R(\boldsymbol{\beta})\big)\,\mathcal{O}'^{i}\,\phi_d\big(R(\boldsymbol{\beta})\big)^{-1} &= \phi_d\big(R(\boldsymbol{\beta})\big)\Big[\big(R(\boldsymbol{\alpha})^{-1}\big)^{i}_{\ j}\mathcal{O}^{j}\Big]\,\phi_d\big(R(\boldsymbol{\beta})\big)^{-1} = \\ &= \big(R(\boldsymbol{\alpha})^{-1}\big)^{i}_{\ j}\phi_d\big(R(\boldsymbol{\beta})\big)\,\mathcal{O}^{j}\,\phi_d\big(R(\boldsymbol{\beta})\big)^{-1} = \big(R(\boldsymbol{\alpha})^{-1}\big)^{i}_{\ j}\,\big(R(\boldsymbol{\beta})^{-1}\big)^{j}_{\ k}\,\mathcal{O}^{k} = \\ &= \big(R(\boldsymbol{\alpha})^{-1}\circ R(\boldsymbol{\beta})^{-1}\big)^{i}_{\ k}\,\mathcal{O}^{k} = \Big(\big(R(\boldsymbol{\beta})\circ R(\boldsymbol{\alpha})\big)^{-1}\Big)^{i}_{\ j}\ \mathcal{O}^{j}\end{aligned}$$

Par contre, si on avait défini

$$\mathcal{O}'^{\,i} = \big(R(\boldsymbol{\alpha})\big)^{i}_{\ j}\ \mathcal{O}^{j}$$

en appliquant une deuxième transformation il faudrait montrer que:

$$\mathcal{O}''^{\,i} = \big(R(\boldsymbol{\beta})\circ R(\boldsymbol{\alpha})\big)^{i}_{\ j}\ \mathcal{O}^{j}$$

mais en fait, on trouve en répétant les mêmes calculs qu'avant:

$$\mathcal{O}''^{\,i} = \big(R(\boldsymbol{\alpha})\circ R(\boldsymbol{\beta})\big)^{i}_{\ j}\ \mathcal{O}^{j}$$

qui viole la loi de composition du groupe.

1.2

En appliquant le résultat précédent on trouve:

$$\exp\big(i\alpha^{i}\psi_d(J_i)\big)\,\mathcal{O}^{j}\,\exp\big(-i\alpha^{i}\psi_d(J_i)\big) = \big(e^{-i\alpha^{i}J_i}\big)^{j}_{\ k}\,\mathcal{O}^{k}$$

en développant au premier ordre en α^i on trouve:

$$\begin{aligned}&\big(\mathbb{1}+i\alpha^{i}\psi_d(J_i)\big)\,\mathcal{O}^{j}\,\big(\mathbb{1}-i\alpha^{i}\psi_d(J_i)\big) = \big(\mathbb{1}-i\alpha^{i}J_i\big)^{j}_{\ k}\mathcal{O}^{k}\\ \Rightarrow\quad &\mathcal{O}^{j}+i\alpha^{i}\big[\,\psi_d(J_i),\mathcal{O}^{j}\,\big] = \mathcal{O}^{j}-\alpha^{i}\epsilon_{ijk}\mathcal{O}^{k}\\ \Rightarrow\quad &\big[\,\psi_d(J_i),\mathcal{O}^{j}\,\big] = i\epsilon_{ijk}\,O^{k}\end{aligned}$$

1.3

Puisque la norme d'un vecteur est invariante sous rotation on peut construire un opérateur invariant (qui commute avec tous les éléments de l'algèbre) en prenant la norme de J_i.

2 Exercice 2

2.1

Puisque deux translations dans deux directions différentes commutent on a que:

$$\big[\, P_\mu, P_\nu \,\big] = 0$$

2.2

$$\exp\big(\tfrac{i}{2}\omega^{\mu\nu}\psi(M_{\mu\nu})\big)\,\psi(P_\alpha)\,\exp\big(-\tfrac{i}{2}\omega^{\mu\nu}\psi(M_{\mu\nu})\big) = \big(e^{-\frac{i}{2}\omega^{\mu\nu}M_{\mu\nu}}\big)_\alpha{}^\beta\psi(P_\beta)$$

En utilisant l'expression pour les générateurs dans la représentation fondamentale:

$$\big(M_{\mu\nu}\big)^\alpha{}_\beta = -i(\eta_{\mu\beta}\delta^\alpha_\nu - \eta_{\nu\beta}\delta^\alpha_\mu) \;\Rightarrow\; \big(M_{\mu\nu}\big)_\alpha{}^\beta = -i(\eta_{\alpha\nu}\delta^\beta_\mu - \eta_{\alpha\mu}\delta^\beta_\nu)$$

en développant au premier ordre en $\omega^{\mu\nu}$ on trouve:

$$\begin{aligned}
&\big(\mathbb{1} + \frac{i}{2}\omega^{\mu\nu}\psi(M_{\mu\nu})\big)\,\psi(P_\alpha)\,\big(\mathbb{1} - \frac{i}{2}\omega^{\mu\nu}\psi(M_{\mu\nu})\big) = \big(\mathbb{1} - \frac{i}{2}\omega^{\mu\nu}M_{\mu\nu}\big)_\alpha{}^\beta\psi(P_\beta)\\
\Rightarrow\quad &\psi(P_\alpha) + \frac{i}{2}\omega^{\mu\nu}\big[\,\psi(M_{\mu\nu}),\psi(P_\alpha)\,\big] = \psi(P_\alpha) - \frac{1}{2}\omega^{\mu\nu}\big(\psi(P_\mu)\eta_{\alpha\nu} - \psi(P_\nu)\eta_{\alpha\mu}\big)\\
\Rightarrow\quad &\big[\, M_{\mu\nu}, P_\alpha \,\big] = i\big(P_\mu\eta_{\alpha\nu} - P_\nu\eta_{\alpha\mu}\big)
\end{aligned}$$

2.3

$$\begin{aligned}
&\exp\big(\tfrac{i}{2}\omega^{\mu\nu}\psi(M_{\mu\nu})\big)\,\psi(M_{\rho\sigma})\,\exp\big(-\tfrac{i}{2}\omega^{\mu\nu}\psi(M_{\mu\nu})\big) = \big(e^{-\frac{i}{2}\omega^{\mu\nu}M_{\mu\nu}}\big)_\rho{}^\alpha\big(e^{-\frac{i}{2}\omega^{\mu\nu}M_{\mu\nu}}\big)_\sigma{}^\beta\psi(M_{\alpha\beta})\\
\Rightarrow\quad &\psi(M_{\rho\sigma}) + \frac{i}{2}\omega^{\mu\nu}\big[\,\psi(M_{\mu\nu}),\psi(M_{\rho\sigma})\,\big] = \psi(M_{\rho\sigma}) - \frac{1}{2}\omega^{\mu\nu}\delta^\alpha_\rho(M_{\mu\nu})_\sigma{}^\beta\psi(M_{\alpha\beta}) -\\
&-\frac{1}{2}\omega^{\mu\nu}\delta^\beta_\sigma(M_{\mu\nu})_\rho{}^\alpha\psi(M_{\alpha\beta})\\
\Rightarrow\quad &i\omega^{\mu\nu}\big[\,\psi(M_{\mu\nu}),\psi(M_{\rho\sigma})\,\big] = -\omega^{\mu\nu}\psi(M_{\rho\beta})\big(\eta_{\nu\sigma}\delta^\beta_\mu - \eta_{\mu\sigma}\delta^\beta_\nu\big) - \omega^{\mu\nu}\psi(M_{\alpha\sigma})\big(\eta_{\nu\rho}\delta^\alpha_\mu - \eta_{\mu\rho}\delta^\alpha_\nu\big)\\
\Rightarrow\quad &\big[\, M_{\mu\nu}, M_{\rho\sigma} \,\big] = i\big(\eta_{\nu\sigma}M_{\rho\mu} - \eta_{\mu\sigma}M_{\rho\nu} + \eta_{\nu\rho}M_{\mu\sigma} - \eta_{\mu\rho}M_{\sigma\nu}\big)
\end{aligned}$$

3 Exercice 3

$$
\begin{aligned}
[\,M_{\mu\nu}, M_{\alpha\beta}\,]f(x) &= -\Big\{(x_\mu\partial_\nu - x_\nu\partial_\mu)(x_\alpha\partial_\beta - x_\beta\partial_\alpha) - \left(\begin{array}{c}\mu\leftrightarrow\alpha\\ \nu\leftrightarrow\beta\end{array}\right)\Big\}f(x)\\
&= -\Big\{x_\mu\partial_\nu(x_\alpha\partial_\beta f) - x_\mu\partial_\nu(x_\beta\partial_\alpha f) - x_\nu\partial_\mu(x_\alpha\partial_\beta f) + x_\nu\partial_\mu(x_\beta\partial_\alpha f) - \left(\begin{array}{c}\mu\leftrightarrow\alpha\\ \nu\leftrightarrow\beta\end{array}\right)\Big\}\\
&= -\Big\{x_\mu\eta_{\alpha\nu}\partial_\beta f + x_\mu x_\alpha\partial_\nu\partial_\beta f - x_\mu\eta_{\nu\beta}\partial_\alpha f - x_\mu x_\beta\partial_\nu\partial_\alpha f\\
&\qquad - x_\nu\eta_{\alpha\mu}\partial_\beta f - x_\nu x_\alpha\partial_\mu\partial_\beta f + x_\nu\eta_{\beta\mu}\partial_\alpha f + x_\nu x_\beta\partial_\mu\partial_\alpha f - \left(\begin{array}{c}\mu\leftrightarrow\alpha\\ \nu\leftrightarrow\beta\end{array}\right)\Big\}\\
&= -\Big\{x_\mu\eta_{\alpha\nu}\partial_\beta f - x_\mu\eta_{\nu\beta}\partial_\alpha f - x_\nu\eta_{\alpha\mu}\partial_\beta f + x_\nu\eta_{\beta\mu}\partial_\alpha f - \left(\begin{array}{c}\mu\leftrightarrow\alpha\\ \nu\leftrightarrow\beta\end{array}\right)\Big\}\\
&= -\Big\{\eta_{\alpha\nu}(x_\mu\partial_\beta - x_\beta\partial_\mu) + \eta_{\mu\beta}(x_\nu\partial_\alpha - x_\alpha\partial_\nu) - \eta_{\beta\nu}(x_\mu\partial_\alpha - x_\alpha\partial_\mu) - \eta_{\alpha\mu}(x_\nu\partial_\beta - x_\beta\partial_\nu)\Big\}f(x)\\
&= i\big\{\eta_{\alpha\nu}M_{\mu\beta} + \eta_{\beta\mu}M_{\nu\alpha} - \eta_{\beta\nu}M_{\mu\alpha} - \eta_{\alpha\mu}M_{\nu\beta}\big\}f(x)
\end{aligned}
$$

Les autres relations de commutation peuvent être trouvées de la même façon.