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Stopping Theorems

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1 Applications of martingale convergence theorem

1.1 Critical percolation on $\mathbb{T}_d$

$p = p_c = \frac{1}{d}$
$X_n = |\delta_n \cap C_0|$
$M_n = \frac{X_n}{(pd)^n} = X_n$ is a martingale.
$X_n \geq 0 \, \forall n$
We need that $\sup \mathbb{E}[M_n^+] < \infty : \sup \mathbb{E}[(-X_n)^+] = 0$, then $X_n \to X$ a.s.
$X_n(\omega) \to X(\omega)$ for almost all ω
$X_n(\omega) = X(\omega)$ for $n \geq N(\omega)$
$\mathbb{P}[X_n(\omega) = k \; \forall n \geq N(\omega)] = \mathbb{P}[X = k] = 0$ by
$\mathbb{P}[X_{n+1} = 2k | X_n k] \geq (p^2)^k > 0$
$\Rightarrow X \equiv 0$ a.s.

1.2 Recurrence of simple random walk on $\mathbb{Z}$

$S_n = \sum_{i=1}^n X_i$ where $X \in \{-1, 1\}$ each with probability $\frac{1}{2}$. Define the stopping time $N = \inf\{n : S_n = -1\}$, the first time we reach -1. $M_n = S_{N \wedge n}$ is a martingale and is bounded below by -1. Since it is a martingale, it is also a super martingale, so $S_{N\wedge n} \xrightarrow{a.s.} S$ as $n \to \infty$, by the martingale convergence theorem.

If $k > -1$ then $\mathbb{P}[M_{n+1} = k | M_n = k] = 0$ so $M_n \to M \equiv -1$. Thus, $M < \infty$ almost surely. Thus, you are bound to return to -1.

Exercise 1. *Show that you can conclude that you will return to -1 infinitely often. Also, show this will hold true for all negative integers and conclude it holds for all integers.*

1.3 Two stopping Theorems

Theorem 1 (Stopping Theorem 1)**.** *Suppose N is a stopping time, such that $\Pr[N \leq k] = 1$ and we have a submartingale $(M_n)_{n\geq 0}$. Then*

$$\mathbb{E}[M_0] \leq \mathbb{E}[M_N] \leq \mathbb{E}[M_k].$$

Proof. $M_{N\wedge n}$ is a submartingale. So we know

$$\mathbb{E}[M_0] = \mathbb{E}[M_{N\wedge 0}] \leq \mathbb{E}[M_{N\wedge k}] = \mathbb{E}[M_N]$$

Let $K_n = 1_{n>N} = 1_{N\leq n-1}$, so we have K_n is predictable. Now, $(K \cdot M)_n = M_n - M_{N\wedge n}$, which is a submartingale. Now,

$$0 = \mathbb{E}[(K \cdot M)_0] \leq \mathbb{E}[(K \cdot M)_k] = \mathbb{E}[M_k] - \mathbb{E}[M_N].$$

□

Exercise 2. *Generalize the above argument to two stopping times (S, T). If $S \leq T$, $\Pr[T \leq k] = 1$, and M_n is a submartingale, then $\mathbb{E}[M_S] \leq \mathbb{E}[M_T]$.*

Theorem 2 (Stopping Theorem 2)**.** *Let M_n be a submartingale such that $(M_n) \leq b$ almost surely. Suppose T is a stopping time with $T < \infty$ almost surely, then*

$$\mathbb{E}[M_0] \leq \mathbb{E}[M_T].$$

Proof. $T < \infty$ gives that for any $\epsilon > 0$ there exists a t_ϵ such that $\Pr[T > t_\epsilon] \leq \epsilon$. Let $S = \min(T, t_\epsilon)$, which is a bounded stopping time. So we have:

$$\begin{aligned}
\mathbb{E}[M_0] &\leq \mathbb{E}[M_S] \text{ (by stopping theorem 1)} \\
&\leq \mathbb{E}[M_T] + 2b \Pr[T > t_\epsilon] \\
&\leq \mathbb{E}[M_T] + 2b\epsilon
\end{aligned}$$

Thus, $\mathbb{E}[M_S] \leq \mathbb{E}[M_T]$ as ϵ is arbitrarily small.

□

2 Gambler's Ruin

We want to know the probability that a gambler will reach ruin. Let $S_n = \sum_{i=1}^{n} X_i$, $T_a = \inf\{n : S_n = -a\}$ the time the gambler reaches ruin and stops playing, and $T_b = \inf\{n : S_n = b\}$ the time the gambler wins enough to be satisfied and stops playing. So the probability of the gambler's ruin is $\Pr[T_a < T_b] = p$.

Let $T = T_a \wedge T_b$. $S_{T\wedge n}$ is a bounded martingale, so we can apply stopping theorem 2. Thus, $\mathbb{E}[S_T] = \mathbb{E}[S_0] = 0$ and $\mathbb{E}[S_T] = p(-a) + (1-p)b \Rightarrow p = \frac{b}{a+b}$.

Now, we want to know $\mathbb{E}[T]$. Let $Y_n = S_n^2 - n$.

Claim 1. *Y_n is a martingale.*

$$\begin{aligned}\mathbb{E}[S_{n+1}^2 - (n+1)|\mathcal{F}_n] &= \frac{1}{2}[(S_n+1)^2 - n - 1] + \frac{1}{2}[(S_n-1)^2 - n - 1] \\ &= (S_n)^2 - n.\end{aligned}$$

$$\begin{aligned}\mathbb{E}[Y_T] &= \mathbb{E}[Y_0] = 0 \\ \mathbb{E}[Y_T] &= \mathbb{E}[S_T^2] - \mathbb{E}[T] \\ \mathbb{E}[S_T^2] &= pa^2 + (1-p)b^2 = ab \\ \mathbb{E}[T] &= ab\end{aligned}$$

Y_n is not bounded, so we cannot use our stopping theorems, but the above still holds true.

3 Optional stopping theorem

Theorem 3 (Optional stopping theorem)**.** *Suppose M_n is a uniformly integrable submartingale and $T < \infty$ is a stopping time. Then*

$$\mathbb{E}[M_0] \le \mathbb{E}[M_T] \le \mathbb{E}[M_\infty].$$

4 Uniform Integrability

Suppose that $\mathbb{E}[|Y|] < \infty$ is integrable. Then $\lim_{c\to\infty} \mathbb{E}[|Y| \cdot 1_{|Y|>c}] \to 0$.

Definition 1 (Uniform integrability)**.** *A collection $(X_i)_{i\in I}$ is uniformly integrable if* $\lim c \to \infty \sup_{i\in I} \mathbb{E}[|X_i| \cdot 1_{|X_i|>c}] = 0$. *This gives $|X_i| \le |Y|$ $\forall i \in I$ for some Y, where $\mathbb{E}[|Y|] < \infty$.*

Exercise 3. *Let ϕ be a function such that $\frac{\phi(x)}{x} \to \infty$ as $x \to \infty$, e.g. $\phi(x) = x^{1+\epsilon}$ or $\phi(x) = x \log^+(x)$. If $\mathbb{E}[\phi(|X_i|)] < c \; \forall i \in I$, then (X_i) is uniformly integrable. Note, that this is necessary for uniform integrability.*

Theorem 4. *The following statements are equivalent.*

- *X_n is a uniformly integrable martingale*
- *$X_n \to X$ almost surely and converges in L^1*
- *There exists r.v. X such that $\mathbb{E}[|X|] < \infty$ and $X_n = \mathbb{E}[X|\mathcal{F}_n]$*

Theorem 5 (Stopping and bounded differences)**.** *Let M_n be a submartingale with $\mathbb{E}(|M_{n+1} - M_n||\mathcal{F}_n) \leq b$. N is a stopping time with $\mathbb{E}[N] < \infty$. Under these assumptions then $M_{n \wedge N}$ is uniformly integrable and $\mathbb{E}[M_0] \leq \mathbb{E}[M_N]$.*

Note that this theorem is a corollary of the optional stopping time theorem.

Proof. Using the triangle inequality

$$|M_{N \wedge n}| \leq |M_0| + \sum_{m=0}^{\infty} |M_{m+1} - M_m| \cdot 1_{m \leq N}.$$

So we want to bound $A = \sum_{m=0}^{\infty} |M_{m+1} - M_m| \cdot 1_{m \leq N}$. Using the tower property

$$\begin{aligned}
\mathbb{E}[A] &= \sum_{m=0}^{\infty} \mathbb{E}[\mathbb{E}[|M_{m+1} - M_m| \cdot 1_{m \leq N} | \mathcal{F}_m]] \\
&\leq \sum_{m=0}^{\infty} b \Pr[N \geq m] \\
&\leq b\mathbb{E}[N] \\
&< \infty
\end{aligned}$$

□

Lemma 6 (Wald's Equation)**.** *Let $S_n = \sum_{i=1}^{n} X_i$, where the X_i's are independent and $\mathbb{E}[X_i] = \mu$, $\mathbb{E}|X_i| < \infty$. If T is a stopping time then $\mathbb{E}[S_T] = \mu\mathbb{E}[T]$.*

Exercise 4. *Prove Wald's Equation.*

5 Asymmetric simple random walk

$S_n = \sum_{i=1}^n X_i$, with $\Pr[X_i = +1] = p > \frac{1}{2}$ and $Pr[X_i = -1] = q = 1 - p$.

Theorem 7. *1.* $\phi(x) = (\frac{1-p}{p})^x \Rightarrow \phi(S_n)$ *is a martingale.*

2. $T_x = \inf\{n : S_n = x\}$ *and* $a < 0 < b$*, then* $\alpha = \Pr[T_a < T_b] = \frac{\phi(b)-\phi(0)}{\phi(b)-\phi(a)}$.

3. If $a < 0$*, then* $\Pr[\min_n S_n \leq a] = (\frac{1-p}{p})^{-a}$.

4. If $b > 0$*, then* $\Pr(T_b < \infty) = 1$ *and* $\mathbb{E}[T_b] = \frac{b}{2p-1}$.

Proof. 1. Left as an exercise to check that the conditions of a martingale are met.

2. Let $T = T_a \wedge T_b$. Claim $T < \infty$ a.s. Proof of the claim is left as an exercise.
$\phi(S_{T\wedge n})$ is a martingale and is also bounded.
$\phi(0) = \mathbb{E}[\phi(S_T)] = \alpha\phi(a) + (1-\alpha)\phi(b)$. Solving for α gives the desired result.

3. $\Pr[\min_n S_n \leq a] = \Pr[T_a < \infty]$. Take $b \nearrow \infty$. Then $\{T_a < \infty\} = \{T_a < T_b\}$ for some b.

4. $\phi(a) \to \infty$ as $a \to -\infty \Rightarrow \Pr[T_b < \infty] = 1$. Now for $\mathbb{E}[T_b] = \frac{b}{2p-1}$.
Let $M_n = S_n - n(p-q)$, which is a martingale. Observe that $T_b \wedge n$ is a bounded stopping time.
$0 = \mathbb{E}[S_{T_b \wedge n} - (T_b \wedge n)(p-q)]$
Observe that $b \geq S_{T_b\wedge n} \geq \inf_n S_n$ and $\Pr(\inf_n S_n) = (\frac{1-p}{p})^{-a}$. So we can bound $\mathbb{E}[S_{T_b\wedge n}]$. We can use a similar argument to bound $\mathbb{E}[T_{b\wedge n}]$. Then using the dominated convergence theorem, we can bring the limit inside.

□