**MASSACHUSETTS INSTITUTE OF TECHNOLOGY**

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# Comparison of Markov Chains

**Content.**

1. Path method
2. Continuous-time M.C.

## 1 Path method

Recall that $\mathcal{E}(f,f) = \frac{1}{2}\mathbb{E}[f(X_1) - f(X_0)^2] = \sum_e c(e)\Delta f(e)^2$, where $c(e) = c(x,y) \triangleq \pi(x)P(x,y)$, $\Delta f(e) = f(y) - f(x)$.

**Proposition 1.** *Let $(\tilde{P}, \tilde{\pi}, \tilde{\mathcal{E}})$ be base M.C. with spectral gap $\tilde{\gamma}$. Suppose $\forall f$, $\mathrm{Var}_\pi(f) \le c\mathrm{Var}_{\tilde{\pi}}(f)$ and $\tilde{\mathcal{E}}(f,f) \le B\mathcal{E}(f,f)$, then*

$$\gamma = \inf_f \frac{\mathcal{E}(f,f)}{\mathrm{Var}_\pi(f)} \ge \frac{1}{BC} \inf_f \frac{\tilde{\mathcal{E}}(f,f)}{\mathrm{Var}_{\tilde{\pi}}(f)} = \frac{\tilde{\gamma}}{BC}$$

**Proposition 2** (V). *Suppose $\frac{\pi(x)}{\tilde{\pi}(x)} \le c < \infty, \forall x$, then $\mathrm{Var}_\pi(f) \le \mathrm{Var}_{\tilde{\pi}}(f)$.*

*Proof.*

$$\begin{aligned}
\mathrm{Var}_\pi(f) + (\mathbb{E}_\pi f - \mathbb{E}_{\tilde{\pi}} f)^2 &= \mathbb{E}_\pi[(f - \mathbb{E}_{\tilde{\pi}} f)^2] \\
&= \sum_x \pi(x)(f(x) - \mathbb{E}_{\tilde{\pi}} f)^2 \\
&\le c \sum_x \tilde{\pi}(x)(f(x) - \mathbb{E}_{\tilde{\pi}} f)^2 \\
&= c\mathrm{Var}_{\tilde{\pi}}(f)
\end{aligned}$$

□

Consider L.R.W. on $P_3$ with $(P, \pi, \mathcal{E})$ and on $C_3$ with $(\tilde{P}, \tilde{\pi}, \tilde{\mathcal{E}})$. $\tilde{P} = \frac{1}{4}I + \frac{1}{4}J$, where $J$ is the matrix with all entries equal to 1. $sp(\tilde{P}) = \{1, \frac{1}{4}, \frac{1}{4}\} \Rightarrow \tilde{r} = \frac{3}{4}$. Since $c(e) = \frac{1}{8}$ and $\tilde{c}(e) = \frac{1}{12}$,

$$\begin{aligned}
\mathcal{E}(f,f) &= \frac{1}{8}\big(\Delta f(e_1)^2 + \Delta f(e_2)^2\big) \\
\tilde{\mathcal{E}}(f,f) &= \frac{1}{12}\big(\Delta f(e_1)^2 + \Delta f(e_2)^2 + \Delta f(e_3)^2\big) \\
&\leq \frac{1}{12}\big(3\Delta f(e_1)^2 + 3\Delta f(e_2)^2\big) \\
&\big(\because \Delta f(e_3) = \Delta f(e_1) + \Delta f(e_2)\big)
\end{aligned}$$

Thus, $\tilde{\mathcal{E}}(f,f) \leq \frac{3 \cdot 8}{12}\mathcal{E}(f,f) = 2\mathcal{E}(f,f)$.

**Proposition 3** ($\mathcal{E}1$)**.** *Suppose that* $\forall \tilde{e} = (x,y)$ *of* $\tilde{P}$*, we choose path* $\Pi_{\tilde{e}} = (e_1, e_2, ...e_k), e_j \in P$ *from* $x$ *to* $y$*. Let*

$$\begin{aligned}
L_{max} &= \max_{\tilde{e}} |\Pi_{\tilde{e}}| \\
\rho &= \max_{e} \frac{1}{c(e)} \sum_{\Pi_{\tilde{e}} \ni e} \tilde{c}(e), \textit{capacity overload}
\end{aligned}$$

*then* $\tilde{\mathcal{E}}(f,f) \leq \rho L_{max}\mathcal{E}(f,f)$*.*

*Proof.*

$$\begin{aligned}
\tilde{\mathcal{E}}(f,f) &\stackrel{\Delta}{=} \sum_{\tilde{e}} \tilde{c}(\tilde{e}) \Delta f(\tilde{e})^2 \\
&\leq \sum_{\tilde{e}} |\Pi_{\tilde{e}}| \sum_{\Pi_{\tilde{e}} \ni e} \Delta f(e)^2 \tilde{c}(\tilde{e}) \\
&\leq L_{max} \sum_{e} \Delta f(e)^2 c(e) \big(\frac{1}{c(e)} \sum_{\Pi_{\tilde{e}} \ni e} \tilde{c}(\tilde{e})\big) \\
&\leq \rho L_{max} \mathcal{E}(f,f)
\end{aligned}$$

where the first inequality results from $\Delta f(\tilde{e})^2 = \big(\sum_{\Pi_{\tilde{e}} \ni e} \Delta f(e)\big)^2 \leq |\Pi_{\tilde{e}}| \sum_e \Delta f(e)^2$ $\square$

**Example 1.** *Consider L.R.W. on $P_n$ and $C_n$ with $P, \tilde{P}$ respectively. Then $c(e) = \frac{1}{4(n-2)}, \tilde{c}(\tilde{e}) = \frac{1}{4}$.*

$$
\begin{aligned}
\rho &= 2 \times \frac{\tilde{c}(\tilde{e})}{c(e)} \approx 2 \\
L_{max} &= n \\
\mathrm{Var}_{\pi}(f) &= Var_{\tilde{\pi}}(f)(2 \pm \frac{1}{n} \\
\Rightarrow \gamma &\geq \frac{c}{n}\tilde{\gamma} = \frac{c}{n^3}(\ddot{\frown})
\end{aligned}
$$

**Example 2.** *L.R.W. on $P_n$ with $P$. Consider $\tilde{P}(x,y) = \pi(y)$. Then $c(e) = \frac{1}{4(n-2)}, \tilde{c}(\tilde{e}) = \frac{1}{4n}$,*

$$
\begin{aligned}
&\tilde{c}(e) = \pi(x)\pi(y) \sim \frac{1}{n^2} \Rightarrow \tilde{\gamma} = 1, \tilde{\mathcal{E}}(f,f) = \mathrm{Var}_{\tilde{\pi}}(f) \\
&\Pi_{(x,y)} = (x, x+1 \ldots, y) \Rightarrow L_{max} = n \\
&\rho = \frac{n^2 \times \frac{1}{n^2}}{1/n} \Rightarrow \gamma \geq \frac{\tilde{\gamma}}{n^2} = \frac{1}{n^2}(\ddot{\smile})
\end{aligned}
$$

**Example 3.** *L.R.W. on hypercube, with $P$. $c(e) = \frac{1}{2n}2^{-n}$. Consider $\tilde{P}(x,y) = \pi(y)$, $\tilde{c}(\tilde{e}) = 2^{-2n}$. Given $(x,y)$, path is selected by flipping bits in the increasing order like*

$$x = 0011 \to 0111 \to 0101 \to 0100 = y$$

*Observe that*

$$e = -t\, bits - \left\{ \begin{array}{c} 0 \\ 1 \end{array} \right\} - s\, bits -, \quad t + s = n - 1.$$

$$
\begin{aligned}
L_{max} &= n, \\
\rho &= \frac{1}{\frac{1}{2n} \cdot 2^{-n}} \cdot 2^{-2n}\, 2^t\, 2^s = n \\
\Rightarrow \gamma &\geq \frac{1}{n^2}(\ddot{-})
\end{aligned}
$$

**Theorem 1.** *Let $P$ be L.R.W. on edge-transitive graph $G$, $x, y \overset{i.i.d.}{\sim} \pi, d(\cdot,\cdot)$ : graph distance. Then*

$$\gamma \geq \frac{1}{2\mathbb{E}[d^2(x,y)]} \geq \frac{1}{2diam(\mathcal{X})},$$

*Proof.* Let $\Pi_{x,y}$ be a random min-distance path in $G$.

$$\begin{aligned}
2\mathrm{Var}(f) &= \mathbb{E}[\big(f(X)-f(Y)\big)^2] \\
&\leq \mathbb{E}|\Pi_{x,y}|\sum_e \mathbb{I}\{e\in\Pi_{x,y}\}\Delta f(e)^2\frac{1}{4|E|}4|E|\;\big(c(e)=\frac{1}{4|E|}\big) \\
&= \sum_e \Delta f(e)^2 c(e)\cdot 4|E|\,\mathbb{E}[d(x,y)\mathbb{I}\{e\in\Pi_{x,y}\}] \\
&= 4\mathcal{E}(f,f)\mathbb{E}[d^2(x,y)]
\end{aligned}$$

□

**Proposition 1** ($\mathcal{E}2$)**.** *Let* $\Gamma$ *be a set of all path on* $P$*,* $F:\Gamma\to\mathbb{R}_+$ *is a* $(P,\tilde{P})$*-flow if*

$$\begin{aligned}
&\sum_{\Pi:x\to y} F(\Pi) = \tilde{c}(x,y) \\
&B(F) = \max_e \frac{1}{c(e)}\sum_{\Pi\ni e}|\Pi|F(\Pi) \leq L_{max}\max_e\frac{F(e)}{c(e)}
\end{aligned}$$

$\Rightarrow \tilde{\mathcal{E}}(f,f)\leq B\mathcal{E}(f,f)$

**Theorem 2.** *If* $P$ *is a L.R.W. on vertex-transitive G,* $x,y\overset{i.i.d.}{\sim}\pi$

$$F(\Pi)=\begin{cases} 0 & ,|\Pi|\neq d(x,y) \\ \frac{1}{\#min-distance}\cdot\frac{1}{n^2} & ,|\Pi|=d(x,y)\end{cases}$$

*then* $\gamma\geq\frac{1}{2\mathbb{E}[d^2(x,y)]deg(G)}$.

## 2 Continuous-time M.C.

Let $\{X_n\}$ be M.C. with $P$, $N_t$: Poisson process, $N_t\sim\mathrm{Pois}(t)$, $W_t\triangleq X_{N_t}$.

**Proposition 4.** *1.* $W_t$ *- M.C. process*

*2.*

$$\begin{array}{llcll}
X & reversible & \Leftrightarrow & W_t & reversible \\
X & irreducible & \Leftrightarrow & W_t & irreducible \\
X & \pi-stat & \Leftrightarrow & W_t & \pi-stat
\end{array}$$

*3.*

$$
\begin{aligned}
H_t(x,y) &\overset{\Delta}{=} \Pr(W_t = y | W_0 = x), H_t = e^{t(P-I)} \\
\Pr(W_t = y) &= \sum_m \Pr(W_t = y, N_t = m) \\
&= \sum_m \Pr(X_m = y) \Pr(N_t = m)
\end{aligned}
$$

**Theorem 3.** *If* $d_{TV}(P_k(x,\cdot),\pi) \leq \epsilon$*,*

1. $k > \frac{2}{\epsilon} \Rightarrow d_{TV}(H_t(x,\cdot) \leq 2\epsilon, \quad t \geq k + \sqrt{\frac{2k}{\epsilon}}$
2. $P(y,y') \geq \delta > 0$*,* $\forall y$ *and* $d_{TV}(H_t(x,\cdot) \leq \epsilon \Rightarrow d_{TV}(P_k(x,\cdot),\pi)$*,* $k \geq t + O_{\epsilon,\delta}(\sqrt{t})$

*Proof.* *1. We know that* $\Pr(W_t \in E | N_t = k) = \Pr(X_k \in E) = \pi(E) + \epsilon$

$$
\begin{aligned}
&\Rightarrow \forall l \geq k, \Pr(W_t \in E | N_t = l) = \pi(E) \pm \epsilon \\
&\Rightarrow \forall t, \Pr(W_t \in E | N_t \geq k) = \pi(E) \pm \epsilon, \\
&\quad t \geq k + c_\epsilon \sqrt{k}, \Pr(N_t \geq k) \geq 1 - \epsilon \\
&\Rightarrow \Pr(W_t \in E) = \pi(E) \pm 2\epsilon
\end{aligned}
$$

*2.* $P = \delta I + (1-\delta)Q$*, let* $\{Y_k\}$ *be a M.C. with* $Q$*, then*

$$
\begin{aligned}
&X_k = Y_{L_k}, W_t = Y_{N_{\bar{\delta}_t}}, L_k = \mathrm{Bin}(k, \bar{\delta}) \\
&W_t - mixed \Rightarrow Y_k - mixed \Rightarrow X_t - mixed
\end{aligned}
$$

□

$$
||H_t f - \mathbb{E}_\pi f||_2^2 \leq e^{-2\gamma t} ||f||_2^2, \chi^2(\nu H_t || \pi) \leq e^{-2\gamma t} \chi^2(\nu || \pi) \ \big(\text{not } \gamma_a!\big).
$$