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First and second moment methods applied to percolation

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1 First moment method

Theorem 1 (Markov's inequality). *Let X be a non-negative random variable. Then*

$$\Pr[X \geq a] \leq \frac{\mathbb{E}[X]}{a}$$

for any $a > 0$.

Corollary 2 (The first moment method). *If $X \geq 0$ is a non-negative integer-valued random variable, then*

$$\Pr[X > 0] \leq \mathbb{E}[X].$$

The proof is left as an exercise.

The first moment method is a simple application of Markov's inequality for integer-valued variables (Corollary 2).

Definition 1. *$G(n, p_n)$ is a distribution over graphs on n vertices where each edge is present independently with probability p_n.*

Theorem 3 (Example). *If $np_n \to 0$, then*

$$\Pr_{G \leftarrow G(n,p_n)}[\exists \textit{ a triangle in } G] \to 0.$$

Proof. Let T be the number of triangles in the graph G. From Corollary 2 we get

$$\begin{aligned}\Pr_{G \leftarrow G(n,p_n)}[\exists \text{ a triangle in } G] &= \Pr[T > 0] \leq \mathbb{E}[T] \\ &= \sum_{1 \leq u < v < w \leq n} \mathbb{E}[1_{(u,v,w) \text{ is a triangle}}] \\ &\leq n^3 p_n^3 \to 0. \end{aligned}$$

$\square$

2 Bond percolation on $\mathbb{Z}^2$

Definition 2 (2-dimensional square lattice). $\mathbb{L}^2 := (\mathbb{Z}^2, \mathbb{E}^2)$ *is the graph with vertex set* $\mathbb{Z}^2$ *and edge set*

$$\mathbb{E}^2 := \left\{(u,v) \mid u, v \in \mathbb{Z}^2 \text{ and } \|u - v\|_1 = 1\right\}.$$

Definition 3. *For any* $0 \leq p \leq 1$, $\mathbb{P}_p$ *is the distribution over subgraphs of* $\mathbb{L}^2$ *where we keep each edge independently with probability* p.

Consider the following physical process. Sample a graph from $\mathbb{P}_p$ where each edge is kept with probability p. Pour water at the origin node $(0,0)$. If there is a neighboring edge (edge is "open"), the water flows across it. The water stops to flow when there are no more open edges. The question that we want to answer is for what values of p we have a positive probability of water flowing out infinitely far away. Clearly, if $p = 0$, then there are no open edges and the water cannot flow. On the other hand, if $p = 1$, then the entire graph $\mathbb{L}^2$ is present and the water can flow to infinity.

Given a graph, we write $x \leftrightarrow y$ if and only if there is an open path (consisting of open edges) connecting x to y. We write $C_x := \{y \in \mathbb{Z}^2 \,:\, y \leftrightarrow x\}$ to denote a random subset of the grid consisting of all vertices y connected to the given vertex x. For the sake of notational simplicity we will write 0 to denote the origin $(0,0)$ of the grid $\mathbb{Z}^2$. Let $\theta(p) := \mathbb{P}_p(|C_0| = +\infty)$ be the probability that the connected component containing the origin 0 has an infinite size. Let $p_c := p_c(\mathbb{L}^2) := \sup\{p : \theta(p) = 0\}$ be the critical probability of C_0 having infinite size. E_∞ denotes the event of having an infinite component somewhere (not necessarily containing the origin). The event E_∞ satisfies the following two properties.

- If the edge probability $p < p_c$ is below the critical probability, then the probability of the event E_∞ is 0:

$$\begin{aligned}\mathbb{P}_c(E_\infty) &= \mathbb{P}_p\left(|C_x| = +\infty \text{ for some } x \in \mathbb{Z}^2\right)\\ &\leq \sum_{x\in\mathbb{Z}^2} \mathbb{P}_p(|C_x| = +\infty) = 0,\end{aligned}$$

where we use the fact that the set $\mathbb{Z}^2$ is countable and that $p < p_c$ to establish the last equality.

- Conversely, if $p > p_c$, then, by Kolmogorov's zero-one law, $\mathbb{P}_p(E_\infty) = 1$. That is, the probability that there is an infinite component *somewhere* is equal to 1 in the case when $p > p_c$.

Exercise 1. *Show that the critical probability satisfies* $p_c(\mathbb{L}^1) = 1$ *for the infinite line graph.*

Claim 1. *The critical probability* $p_c(\mathbb{L}^2)$ *of the two-dimensional grid is strictly between* 0 *and* 1.

The following two propositions prove the claim.

Proposition 4. $p_c \geq 1/3$.

Proof. We want to reason about the event that $|C_0| = +\infty$. Our goal is to show that if $p < 1/3$, then $\theta(p) = \mathbb{P}_p(|C_0| = +\infty) = 0$.

Observe that C_0 is of infinite size if and only if there exists a self-avoiding path (SAP) starting from the origin of infinite length consisting of open edges. Let X_n denote the number of self-avoiding paths of at least n open edges, which starts from the origin. Then

$$\begin{aligned}\theta(p) &= \Pr[X_n > 0\ \forall n]\\ &\leq \Pr[X_n > 0 \text{ for some arbitrary and fixed } n]\\ &\leq \mathbb{E}[X_n]\\ &= \sum_{\text{path } p_n \text{ starting at 0 of at least } n \text{ edges}} \Pr[\text{path } p_n \text{ is open}]\\ &\leq 4 \cdot 3^{n-1} \cdot p^n \to 0\end{aligned}$$

if $p < 1/3$. In the last inequality we use the union bound over all paths of length n (not necessarily self-avoiding). The number of such paths is $4 \cdot 3^{n-1}$ since there are 4 possible edges to leave the origin and 3 possible edges to make any further choice. $\square$

The proof of the second proposition uses the dual lattice.

Definition 4 (Dual lattice)**.** *Let $\mathbb{L}^2$ be the two dimensional square lattice. The dual lattice $\widehat{\mathbb{L}}^2$ has vertex set $(1/2, 1/2) + \mathbb{Z}^2$ and the edge set contains edge (u, v) if and only if $\|u - v\|_1 = 1$.*

Definition 5 (Dual lattice and bond percolation)**.** *In the context of bond percolation, suppose we produce first closed and open edges in $\mathbb{L}^2$. Then this naturally produces closed and open edges in $\widehat{\mathbb{L}}^2$; we will call an edge e of the dual lattice open if it crosses an open edge in the original $\mathbb{L}^2$. If (u, v) is crossing a closed edge, we will call it* closed*.*

We will also need the following lemma.

Lemma 5. *C_0 is of infinite size if and only if there is no closed cycle (consisting of closed edges) in the dual lattice $\widehat{\mathbb{L}}^2$ containing the origin.*

The proof is non-trivial and we will not prove it here. The approach of using the above lemma is known as "Peierls' argument". It was developed to show a phase transition behavior in the Ising model in two dimensions or more.

Proposition 6. $p_c < 1$.

Proof. Let M_n be the number variable that is equal to the number of closed cycles of length n in the dual lattice containing the origin. We observe that $\mathbb{E}[M_n] \leq (1-p)^n \cdot \frac{n}{2} \cdot 3^n$ since the number of cycles in the dual lattice containing origin is upper bounded by $\frac{n}{2} \cdot 3^n$ and the probability that all n edges appearing in the cycle are closed is $(1 - p)^n$.

Our goal is to show that $\theta(p) > 0$ for a constant $p < 1$ that is sufficiently close to 1. We get

$$\begin{aligned}
1 - \theta(p) &= \Pr[C_0 \text{ is of finite size}] \\
&\leq \Pr[M_n > 0 \text{ for some } n \geq 4] \text{ (using that the grid is bi-partite)} \\
&\leq \sum_{n=4}^{+\infty} \Pr[M_n > 0] \\
&\leq \sum_{n=4}^{+\infty} \mathbb{E}[M_n] \\
&\leq \sum_{n=4}^{+\infty} (1 - p)^n \cdot \frac{n}{2} \cdot 3^n \\
&= \sum_{n=4}^{+\infty} \frac{n}{2} \cdot [3(1 - p)]^n < 1
\end{aligned}$$

for some sufficiently large constant $p < 1$. $\square$

Exercise 2. *Prove that $p_c \leq 2/3$ via similar arguments as in the above proof. (Hint: let A_N be the event that all edges in an open ball around the origin of radius N are open. For any finite N we have $\Pr[A_N] > 0$. Conditional on A_N, one may consider the summation over closed cycles in the dual lattice starting from a value $\Omega(N^2)$).*

3 Second moment method

Theorem 7 (Chebyshev's inequality)**.** *Let X be a random variable. Then*

$$\Pr[|X - \mathbb{E}[X]| \geq a] \leq \frac{\mathrm{var}[X]}{a^2}$$

for any $a > 0$.

Corollary 8. *If $X \geq 0$ is non-negative random variable, then*

$$\begin{aligned}\Pr[X > 0] &= 1 - \Pr[X = 0] \\ &\geq 1 - \frac{\mathrm{var}[X]}{(\mathbb{E}[X])^2}.\end{aligned}$$

Proof. We use Chebyshev's inequality with $a = \mathbb{E}[X]$. $\square$

Theorem 9 (Second moment method)**.** *For any non-negative but not identically equal to 0, random variable $X \geq 0$ we have*

$$\Pr[X > 0] \geq \frac{(\mathbb{E}[X])^2}{\mathbb{E}[X^2]}.$$

The second moment method is a technique to show that a random variable has a positive probability of being positive. Theorem 9 is an example. It can be proved directly, or observed as a consequence of the following inequality with $\theta \to 0$.

Theorem 10 (Paley-Zygmund inequality)**.** *Let $X \geq 0$ be a non-negative random variable. For any $0 < \theta < 1$ we have*

$$\Pr[X > \theta \cdot \mathbb{E}[X]] \geq (1 - \theta)^2 \frac{(\mathbb{E}[X])^2}{\mathbb{E}[X^2]}.$$

Proof. From Cauchy-Schwarz inequality we get

$$\begin{aligned}\mathbb{E}[X] &= \mathbb{E}[X \cdot 1_{X \leq \theta \cdot \mathbb{E}[X]}] + \mathbb{E}[X \cdot 1_{X > \theta \cdot \mathbb{E}[X]}] \\ &\leq \theta \cdot \mathbb{E}[X] + \sqrt{(\mathbb{E}[X])^2 \Pr[X > \theta \cdot \mathbb{E}[X]]}.\end{aligned}$$

Rearranging gives the required inequality. □

Exercise 3. *Show that Paley-Zygmund inequality is a strict improvement over Chebyshev's inequality in some setting.*

The second moment method can be thought as a converse to the union bound.

Theorem 11. *Consider m events $A_1^m, \dots, A_m^m$. Let $B_m := A_1^m \cup \dots \cup A_m^m$ be the union of the m events and $\mu_m := \sum_{i=1}^m \Pr[A_i^m]$ be the expected number of realized events. We write $i \sim j$ if and only if events A_i^m and A_j^m are* dependent. *We set $\gamma_m := \sum_{i \sim j} \Pr(A_i^m \cap A_j^m)$ to be a measure of the dependence of the m events. The following two statements hold.*

- $\lim \Pr(B_m) > 0$ *if $\mu_m \to +\infty$ and $\gamma \leq C \cdot \mu_m^2$ for some constant $C > 0$.*
- $\lim \Pr(B_m) \to 1$ *if $\mu_m \to +\infty$ and $\gamma = o(\mu_m^2)$.*

The proof is left as an exercise.

4 Percolation on the d-ary tree

We call an infinite tree d-ary if every vertex has exactly d children. We denote this tree by $\widehat{\mathbb{T}}_d$ (we use the hat symbol since $\mathbb{T}_d$ is reserved for d-regular tree). $\mathbb{P}_p$ is the distribution over subgraphs of $\widehat{\mathbb{T}}_d$ where we keep each edge independently with probability p. As before we call an edge "open" if we keep it in the subgraph. C_0 is the connected component containing the origin (root of the tree). $\theta(p)$ is the probability that C_0 has an infinite size. Similarly as before, $p_c := p_c(\widehat{\mathbb{T}}_d) := \sup\{p : \theta(p) = 0\}$.

Theorem 12. *$p_c = 1/d$.*

Proof. We will first show that $p_c \geq 1/d$. Assume that $p < 1/d$. We want to show that the probability that C_0 is infinite is 0. Let ∂n be the set of nodes at the n-th level of the tree (root is at the 0-th level). Let $X_n := |C_0 \cap \partial n|$ be the number of nodes of the connected component C_0 in the n-th level. If $p < 1/d$, then

$$\begin{aligned}\theta(p) = \Pr[|C_0| = +\infty] &= \Pr[X_n > 0\, \forall n] \\ &\leq \Pr[X_n > 0] \text{ (for any particular } n) \\ &\leq \mathbb{E}[X_n] \\ &= d^n \cdot p^n \to 0.\end{aligned}$$

In the last equality we use that fact that $|\partial n| = d^n$ and that we keep each edge on the root-to-node path with probability p.

In the rest we show that $p_c \leq 1/d$. We want to show that if $p > 1/d$, then $\theta(p) > 0$. We will use Theorem 9. We write $\mu_n := \mathbb{E}[X_n] = \mathbb{E}[|C_0 \cap \partial n|] = d^n \cdot p^n \to +\infty$. We want to upper bound $\mathbb{E}[X_n^2]$. Given two nodes x and y in the tree we write MRCA(x, y) to denote their most recent common ancestor.

$$\begin{aligned}\mathbb{E}[X_n^2] &= \mathbb{E}\left[\sum_{x \in \partial n} 1_{0 \leftrightarrow x}\right]^2 \\ &= \sum_{x,y \in \partial n} \Pr[x \in C_0,\ y \in C_0] \\ &= \sum_{x \in \partial n} \Pr[x \in C_0] \ + \sum_{x \neq y \in \partial n} \Pr[x \in C_0,\ y \in C_0] \\ &= \mu_n \ + \sum_{x \neq y \in \partial n} \sum_{m=0}^{n-1} 1_{\text{MRCA}(x,y) \in \partial m} \Pr[x \in C_0,\ y \in C_0] \\ &= \mu_n + \sum_{m=0}^{n-1} p^{2n-m} \cdot d^n \cdot (d-1) d^{n-m-1} \\ &\leq \mu_n + p^{2n} d^{2n} \sum_{m=0}^{+\infty} (pd)^{-m} \\ &= \mu_n + \mu_n^2 \frac{1}{1 - (pd)^{-1}}.\end{aligned}$$

In the second to last inequality we use the fact that there are d^n options for x and $(d-1)d^{n-m-1}$ options for y such that MRCA$(x, y) \in \partial m$. For such a choice of x and y we want that $2n - m$ edges are open which happens with probability p^{2n-m}.

We can use Theorem 9 since

$$\frac{(\mathbb{E}[X_n])^2}{\mathbb{E}[X_n^2]} \geq \left(\frac{1}{\mu_n} + \frac{1}{1 - (pd)^{-1}}\right)^{-1} \geq C_{p,d} > 0.$$

We get

$$\theta(p) = \Pr[X_n > 0 \, \forall n] \geq \lim_{n \to +\infty} \Pr[X_n > 0] \geq C_{p,d} > 0$$

as required. $\square$

Exercise 4. *Show the same statement for d-regular trees.*