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KKL theorem

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1 KKL theorem and some implications

Recall from last time:

1. $I_i(p) := \mathbb{P}_{\mu_p}[f(x) \neq f(x \oplus e_i)]$.
2. For a monotone set Ω, $\frac{d}{dp}\mu_p(\Omega) = \pm I^p(1_\Omega)$, where $I^p(f) := \sum_{i=1}^n I_i^p(f)$.

We consider some implications and corollaries of the following Theorem.

Theorem 1 (Khan-Kalai-Linial)**.** *There exists a constant $c > 0$ such that for any $p \in (0,1)$, $n \in \mathbb{N}$, and $f : \{0,1\}^n \to \{0,1\}$,*

$$I^p(f) \geq c \cdot var_{\mu_p}(f) \log\left(\frac{1}{\tau_p}\right),$$

where $\tau_p := \max_{i \in [n]} I_i^p(f)$.

The proof of the theorem is given in the next lecture. Note that in Section 2 we show it sufficient to consider the case when $p = \frac{1}{2}$.

Corollary 2. *There exists a constant $c > 0$ such that for all $p \in (0,1)$ and $f : \{0,1\}^n \to \{0,1\}$, there exists $i^* \in [n]$ such that*

$$I_{i^*}^p(f) \geq c\frac{\log n}{n} var_{\mu_p}(f).$$

Proof. From the KKL theorem (Theorem 1),

$$n\tau_p \geq c' \cdot var_{\mu_p}(f) \log\left(\frac{1}{\tau_p}\right),$$

which is equivalent to the condition (for some constant $c > 0$),

$$\tau_p \geq c \cdot var_{\mu_p}(f) \frac{\log n}{n},$$

and the desired result follows. □

Implications:

1. The following result is a corollary of the KKL theorem for boolean functions that are monotone and symmetric. See section 9.6 in the book (Boucheron, Lugosi, and Massart) for further similar results.

   Corollary 3. *Let $\Omega \subseteq \{0,1\}^n$ be monotone, $\Psi(p) := \mu_p(\Omega)$, and $p_\epsilon = \Psi^{-1}(\epsilon)$ and $p_{1-\epsilon} = \Psi^{-1}(1-\epsilon)$. Suppose that $I_i^p(1_\Omega) = I_j^p(1_\Omega)$ for all $i, j \in [n]$. Then, there exists a constant c_ϵ such that,*

   $$p_{1-\epsilon} - p_\epsilon \leq \frac{c_\epsilon}{\log n}.$$

   Proof. Take $p \in (p_{1-\epsilon}, p_\epsilon)$. Then, $var_{\mu_p}(1_\Omega) \in (\delta_\epsilon, 1-\delta_\epsilon)$ for some $\delta_\epsilon > 0$. By the KKL theorem, there exists $i^* \in [n]$ and $c_\epsilon > 0$ such that $I_{i^*}^p(f) \geq c_\epsilon \frac{\log n}{n}$. So by symmetry hypothesis,

   $$\frac{d}{dp}\mu_p(1_\Omega) = I^p(1_\Omega) \geq c_\epsilon \log n.$$

   The desired result follows from the fact the above inequality holds for all $p \in (p_\epsilon, 1-p_\epsilon)$. □

   Example : Consider Tribes (see page 269 in the book). Take $X_i \sim \text{Ber}(p)$. Let $\Omega = \{\text{No open path in } G \text{ between end vertices}\}$. By Corollary 3, $p_{1-\epsilon} - p_\epsilon \leq \frac{c_\epsilon}{\log n}$. Note Tribe's function implies KKL is tight (up to constants). We tried to compare with Mergulis' Theorem but realized that we did not satisfy the edge connectivity requirement. Nonetheless, Tribes shows Corollary 2 is tight up to constant factors (see the discussion in the book).

2. We use the KKL theorem (and some other results) to show that the critical threshold for $\mathbb{Z}^2$-edge-perculation is $p_c = \frac{1}{2}$. Recall $\Theta(p) = \mathbb{P}_{\mu_p}[0 \in$ infinite component$]$. Let $A_{n,m} \subset \mathbb{Z}^2$ be an $n \times m$ "block" of $\mathbb{Z}^2$. Define

$$\Psi_{n,m}(p) := \mathbb{P}_{\mu_p}[\exists \text{ open path from left to right side of } A_{n,m}].$$

Theorem 4 (Russo-Seymour-Welsh)**.** *For all $a, b > 0$, there exists $\epsilon(a,b) > 0$ such that for all $n \in \mathbb{N}$,*

$$\epsilon(a,b) \le \Psi_{an,bn}(\frac{1}{2}) \le 1 - \epsilon(a,b).$$

Corollary 5. $\Theta\left(\frac{1}{2}\right) = 0.$

Proof sketch. Recall that 0 is in the infinite component if and only if we cannot construct a cycle (with open edges) surrounding 0 in the dual lattice. Consider "surrounding" zero with four $4n \times n$ rectangular grids. The RSW theorem implies each of these events occurs with at least constant probability for all n. The FKG inequality implies the intersection of the events occurs with at least constant probability. We can construct such "box enclosures" for all n such that they are disjoint. So with probability one at least one event happens. Thus, with probability one, 0 is not in the infinite component. □

Proposition 6. *If $\Psi_{n_0,2n_0}(p) \ge .98$, then $\Psi_{2^k n_0, 2^{k+1} n_0}(p) \ge 1 - \frac{0.02}{2^k}$ and $\Theta(p) > 0$.*

Proof sketch. The proof of the first claim follows by an appropriate "tiling" of blocks and various applications of the union bound. The second claim follows from an appropriate assembly of the $2^k n_0 \times 2^{k+1} n_0$ blocks. □

We get to use the KKL theorem in the proof of the following theorem.

Theorem 7 (Kesten)**.** $p_c = \frac{1}{2}$.

Proof. By Corollary 5, $p_c \ge \frac{1}{2}$. Suppose $p_c > \frac{1}{2}$. Take $\frac{1}{2} < p < p_c$. For all $n \in \mathbb{N}$,

$$\epsilon \le \Psi_{n_0,2n_0}(\frac{1}{2}) \le \Psi_{n_0,2n_0}(p) \le 0.98.$$

where the first inequality follows from RSW, the second from monotonicity and the third from Proposition 6. Let $\mu_p(\Omega_n) = \Psi_{n_0,2n_0}(p)$, where $n = 2n_0^2$. The KKL theorem implies,

$$\frac{d}{dp}\mu_p(\Omega_n) = I_p(\Omega_n) \geq c \log\left(\frac{1}{\tau}\right).$$

But $\tau \to 0$ as $n \to \infty$, which implies $\frac{d}{dp}\mu_p(\Omega_n) \to \infty$ as $n \to \infty$. This yields a contradiction. □

2 Reduction of KKL theorem to $p = \frac{1}{2}$

We briefly show is sufficient to prove Theorem 1 for $p = \frac{1}{2}$. Let $\tilde{X}_i \sim \text{Ber}(p)$ and consider $f(\tilde{X}_1, ..., \tilde{X}_n)$. Assume p has a finite decimal expansion. Then, there exist m and N such that

$$X_i = 1_{\left\{\sum_{j=1}^m 2^{-j} X_{i,j} \geq \frac{N}{2^m}\right\}} =: h(X_{i,1}, ..., X_{i,m}).$$

Define $g(X_{1,1}, ..., X_{n,m}) = f(h(X1, 1, ..., X_{1,m}), ..., h(X_{n,1}, ..., X_{n,m}))$. Note that $var_{\mu_{\frac{1}{2}}}(g) = \text{var}_\mu(f)$, $I^{\frac{1}{2}}(g) \leq 2I^p(f)$, and $I_{\max}(g) \leq \frac{1}{2}I_{\max}(f)$. Thus, it is sufficient to consider $p = \frac{1}{2}$.

3 Fourier decomposition and hypercontractivity

We will use the topics in this section in the proof of the KKL theorem. We first consider the Fourier decomposition for a boolean function $f : \{0,1\}^n \to \{0,1\}$,

$$f(x_1, ..., x_n) = \sum_{S \subseteq [n]} \hat{f}(S) u_S(x_1, ..., x_n),$$

where u_S are orthogonal functions under $\mu_{\frac{1}{2}}$ defined by $u_S(x) = \prod_{i \in S}(-1)^{X_i}$. Note we can derive with the Efron-Stein decomposition. Some properties of the Fourier decomposition are as follows:

1. $\mathbb{E}f^2 = \sum_S \hat{f}^2(S)$.
2. $\text{var}(f) = \sum_{S \neq \emptyset} \hat{f}^2(S)$
3. $I_i^{\frac{1}{2}}(f) = \mathbb{E}|\nabla_i f|^2 = \mathbb{E}\left(\sum_S \hat{f}^2(S)\nabla_i u_S\right)^2 = 4\sum_{S : i \in S} \hat{f}^2(S)$.
4. $\mu_S(x) = \prod_{i \in S}(-1)^{x_i} = \prod_{i \in S}(1 - 2x_i)$.

5. $I(f) = 4\sum_S |S| \hat{f}^2(S) \geq 4\sum_{S \neq \emptyset} \hat{f}^2(S) = 4\text{var}(f)$

6. $\nabla_i u_S(x) = \begin{cases} 0, & i \notin S \\ 2u_S, & i \in S. \end{cases}$

We will also need the following hyperconstractivity inequality, which we will prove next time. For all $t \geq 0$,

$$\sum_S e^{-2t|S|} \hat{f}^2(S) \leq \left(\mathbb{E}|f|^q\right)^{\frac{2}{q}},$$

where $q = 1 + e^{-2t}$.