MASSACHUSETTS INSTITUTE OF TECHNOLOGY

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1 Hypercontractivity proof

Theorem 1. *Let* $f : \{0,1\}^n \to \mathbb{R}$ *and let*

$$T_t f(x) \triangleq \sum_{y\in\{0,1\}^n} f(y) \left(\frac{1-e^{-t}}{2}\right)^{\|y-x\|_H} \left(\frac{1+e^{-t}}{2}\right)^{n-\|y-x\|_H} ,$$

then

$$\|T_t f\|_2 \le \|f\|_q, \qquad q = 1 + e^{-2t} ,$$

where $\|f\|_p = \left(\sum_{x\in\{0,1\}^n} |f(x)|^p 2^{-n}\right)^{\frac{1}{p}}$.

Proof. Check for $n = 1$ is routine and then we apply the induction lemma. □

Lemma 1. *If* $A_{i,j} \ge 0$ *and* $B_{k,l} \ge 0$ *are two matrices and for any* f, g *and* $0 < q < 2$ *we have*

$$\|Af\|_2 \le \|f\|_q, \qquad \|Bg\|_2 \le \|g\|_q ,$$

then for any h

$$\|A \otimes Bh\|_2 \le \|h\|_q ,$$

where $\|f\|_p = \left(\sum_{x\in\{0,1\}^n} |f_j|^p\right)^{\frac{1}{p}}$

Remark: Lemma also holds for weighted p-norms, i.e. $\|f\|_p^p = \sum_j \mu(j)|f(j)|^p$, but for simplicity we only consider unweighted sums, which is sufficient for the theorem above.

Proof. WLOG, $h_{j,l} \ge 0$ and we have to prove

$$\left(\sum_{i,k}\left(\sum_{j,l} A_{i,j}B_{k,l}h_{j,l}\right)^2\right)^{\frac{1}{2}} \le \left(\sum_{j,l} h_{j,l}^q\right)^{\frac{1}{q}} . \tag{1}$$

Let $\tilde{h}_{k,j} \triangleq \sum_l B_{k,l} h_{j,l}$, then from statement about A we have for each k:

$$\sum_i \left(\sum_j A_{i,j} \tilde{h}_{k,j} \right)^2 \leq \left(\sum_j |\tilde{h}_{k,j}|^q \right)^{\frac{2}{q}} .$$

Summing this over k and taking the square root we get

$$\left(\sum_{i,k} \left(\sum_j A_{i,j} \tilde{h}_{k,j} \right)^2 \right)^{\frac{1}{2}} \leq \left(\sum_k \left(\sum_j |\tilde{h}_{k,j}|^q \right)^{\frac{2}{q}} \right)^{\frac{1}{2}} .$$

We now recall that by Minkowski inequality we can interchange the order of q and 2 norms to get a larger quantity, so that we continue as

$$\leq \left(\sum_j \left(\sum_k |\tilde{h}_{k,j}|^2 \right)^{\frac{q}{2}} \right)^{\frac{1}{q}} . \tag{2}$$

Finally, from the statement about B we have for every j:

$$\left(\sum_k |\tilde{h}_{k,j}|^2 \right)^{\frac{1}{2}} \leq \left(\sum_l |h_{j,l}|^q \right)^{\frac{1}{q}} ,$$

and substituting into (2) we obtain

$$(2) \leq \left(\sum_{j,l} |h_{j,l}|^q \right)^{\frac{1}{q}} ,$$

which is (1). $\square$