MASSACHUSETTS INSTITUTE OF TECHNOLOGY

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Spatial and Temporal Mixing of 2D Ising Model and Correlation Inequality

Content.

1. Ising model: spatial mixing $\Leftrightarrow$ temporal mixing
2. Correlation inequality

1 Ising model: spatial mixing $\Leftrightarrow$ temporal mixing

Recall that the Ising model is defined as:

$$G = (V, E), x \in \{-1, +1\}^V, \mu(x) = \frac{1}{Z} \exp\left(\beta \sum_{i,j \in E} x_i x_j\right),$$

and the Glauber dynamic is

- Choose $u \in V$ u.a.s.
- Update x_u according to $\Pr[x_u = + | x_{-u}] = \frac{\exp\left(2\beta \sum_{w \in N(u)} x_w\right)}{1+\exp\left(2\beta \sum_{w \in N(u)} x_w\right)}$.

We define the conditional measure: $W \subseteq V, \eta \in \{-1, +1\}^W$, then $\pi_\eta(\cdot) = \Pr[x \in \cdot | x_W = \eta]$.

Theorem 1. *Suppose that for any $\eta \in \{-1, +1\}^W$ and some $\alpha > 0$,*

$$\left|\pi_\eta^{i=+}(x_0 = +) - \pi_\eta^{i=-}(x_0 = +)\right| \leq \exp(-\alpha d(i, 0)),$$

then the Glauber dynamic on G has $t_{mix} = O(n \log n)$ for any boundary condition.

Definition 1 (Block dynamics)**.** *Fix some size L. Block dynamics is a Markov chain on 2D Ising models such that at each step, an $L \times L$ block is chosen uniformly at random and updated conditional on the boundary.*

Theorem 2. *If block dynamic contract $1 - \frac{L^2}{n}$ implies Glauber dynamics mixes in $O(n \log n)$.*

Proof of Theorem 1. Goal: fast mixing for block dynamics.

Start with $x, y \in \{-1, +1\}^V$, such that $x_i \neq y_i, x_j = y_j \ \forall j \neq i$. Find a coupling $X_1, Y_1 \sim \Pr[x, \cdot], \Pr[y, \cdot]$, and bound $\mathbb{E}[\rho(X_1, Y_1)]$.

Coupling:

- Choose the same $L \times L$ block.
- If boundary is same, do same update.
- If boundary is different, update in a monotone way: if $x_0(i) = -1, y_0(i) = +1$, then $X_1(j) \leq Y_1(j)$ for all j in box.

Proof that $Y_1(j) \geq X_1(j)$ a.s. is possible. Run Glauber dynamics in side block, $\tilde{X}_{t+1} \leq \tilde{Y}_{t+1}$ if $\tilde{X}_t \leq \tilde{Y}_t$.

Lemma 3. $\Pr[X_1(j) \neq Y_1(j)] = \Pr[Y_1(j) = +1] - \Pr[X_1(j) = +1]$.

Proof.

$$\begin{aligned}
\Pr[Y_1(j) = +1, X_1(j) = -1] &= \Pr[Y_1(j) = +1] - \Pr[Y_1(j) = +1, X_1(j) = +1] \\
&= \Pr[Y_1(j) = +1] - \Pr[X_1(j) = +1]
\end{aligned}$$

□

Bounding $\mathbb{E}[\rho(X_1, Y_1)]$

1. If box contains i, $\rho(X_1, Y_1) = 0$.
2. If i is on boundary of box, then there are $4L$ choices of box.

$$\begin{aligned}
\mathbb{E}[\rho(X_1, Y_1)] &= \sum_{j \in \text{box}} \Pr[X_1(j) \neq Y_1(j)] \\
&= \sum_{j \in \text{box}} \Pr[Y_1(j) = +1] - \Pr[X_1(j) = +1] \\
&= \sum_{j \in B} 1 + \sum_{1 \in B^C} \exp(-\alpha c \sqrt{L}) \\
&\leq c^2 L + L^2 \exp(-\alpha c \sqrt{L}).
\end{aligned}$$

Given $\epsilon > 0$, we take c small and L sufficiently big such that the above expression is $\leq \epsilon L$.

Combine the above, one has

$$\mathbb{E}[\rho(X_1, Y_1)] \leq 1 - \frac{L^2}{n} + \frac{4L}{n}\epsilon L = 1 - \frac{c' L^2}{n}.$$

Path coupling $\Rightarrow$ block dynamics mixes fast. $\square$

2 Correlation inequality

Claim 1. *Any two nodes in the Ising model are positively correlated,* $\mathbb{E}[X_i X_j] \geq \mathbb{E}[X_i]\mathbb{E}[X_j]$.

Definition 2. *A measure μ on a poset $(\Omega = \{-1, +1\}^V, x \leq y \Leftrightarrow x_i \leq y_i)$, is positively correlated if for increasing f, g, and $X \sim \mu$,*

$$\mathbb{E}[f(X)g(X)] \geq \mathbb{E}[f(X)]\mathbb{E}[g(X)].$$

Theorem 4 (Chebyshev)**.** *If Ω is totally ordered, then any μ has positive correlation.*

Proof. Suppose f, g are increasing, $f(x) \leq f(y) \Leftrightarrow g(x) \leq g(y)$. then

$$\begin{aligned}
&\int\int [f(x) - f(y)][g(x) - g(y)] d\mu(x) d\mu(y) \geq 0 \\
\Rightarrow &\mathbb{E}[f(x)g(x)] - \mathbb{E}[f(x)]\mathbb{E}[g(x)] \geq 0.
\end{aligned}$$

$\square$

Theorem 5 (Harris)**.** *Any product measure on coordinates that are totally ordered has positive correlations (product space is a poset with "$\leq$" obtained from coordinate-wise order).*

Proof. Assume μ_i has positive correlation on $\Omega_1 \times \Omega_2, \mu = \mu_1 \times \mu_2$, and f, g increasing on $\Omega_1 \times \Omega_2$, then

$$\begin{aligned}
&\int\int f(x, y) g(x, y) d\mu_1(x) d\mu_2(y) \\
\geq &\int \left[\int f(x, y) d\mu_1(x) \int g(x, y) d\mu_1(x)\right] d\mu_2(y) \\
\geq &\int\int f d\mu_1 d\mu_2 \int\int g d\mu_1 d\mu_2.
\end{aligned}$$

$\square$

Theorem 6 (FKG inequality)**.** *Let $\Omega = \{0,1\}^S$ be a finite poset, μ is a positive probability measure. If $\mu(w_1 \wedge w_2)\mu(w_1 \vee w_2) \geq \mu(w_1)\mu(w_2)$, then μ has positive correlation. Here $w_1 \vee w_2 = \max\{w_1, w_2\}, w_1 \wedge w_2 = \min\{w_1, w_2\}$.*

Remark. *FKG condition is log-supermodular.*

Definition 3 (Submodular)**.** *h is submodular iff for all $S, T \in P(x)$ (power set of x),*

$$h\left(S \cup T\right) + h\left(S \cap T\right) \leq h\left(S\right) + h\left(T\right),$$

or for all $W \subseteq U$,

$$h\left(U + \{e\}\right) - h\left(U\right) \leq h\left(W + \{e\}\right) - h\left(W\right).$$

Example : [Ising model] One can check that if $x_j \geq y_j \; \forall j$, then

$$\frac{\mu\left(x^{i=+1}\right)}{\mu\left(x^{i=-1}\right)} \geq \frac{\mu\left(y^{i=+1}\right)}{\mu\left(y^{i=-1}\right)}.$$

Also,

$$\exp\left(2\beta \sum_{u \in N(i)} x(u)\right) \geq \exp\left(2\beta \sum_{u \in N(i)} y(u)\right).$$

$\Rightarrow$ Ising has positive correlation!

Claim 2. $\Omega = \{0,1\}^S$ *+ FKG condition $\Rightarrow$ if $x(u) \geq y(u)$ $(u \neq v)$, then*

$$\mu(w(v) \in +|w(u) = x(u), u \neq v) \overset{\text{stochastic dominates}}{\geq} \mu(w(v) \in +|w(u) = y(u), u \neq v).$$

Proof. FKG condition $\Rightarrow$

$$\frac{\mu\left(x^{v=1}\right)}{\mu\left(x^{v=0}\right)} \geq \frac{\mu\left(y^{i=1}\right)}{\mu\left(y^{i=0}\right)}.$$

Note also that $\frac{a}{1-a} \geq \frac{b}{1-b} \Rightarrow a \geq b$, we conclude the statement. $\square$