MASSACHUSETTS INSTITUTE OF TECHNOLOGY

6.265/15.070J Lecture 21 May 3, SP17

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The 2D Ising model

Content.

1. Magnetization phase transition and Peierl's argument
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1 Magnetization Phase Transition

Recall that the Ising model on a graph $G = (V, E)$ with inverse temperature β and no external field has distribution

$$\begin{aligned}\mu(x) &= \frac{1}{Z}\exp\left(\beta \sum_{(i,j)\in E} x_i x_j\right) \\ &= \frac{1}{Z}\exp\left(\beta \cdot \#\text{agreements} - \beta \cdot \#\text{disagreements}\right)\end{aligned}$$

for each $x \in \{-1, 1\}^n$ where $n = |V|$. In this lecture, we will consider the 2D Ising model on the graph $\mathbb{L}^2 \cap [0, \sqrt{n}]^2$ which is a $\sqrt{n} \times \sqrt{n}$ unit lattice where two vertices are adjacent if and only if they are at distance exactly one. We also in general assume no external field i.e. $h = 0$. Recall that the magnetization of an Ising model is

$$\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

Let β_c denote the critical temperature for this 2D Ising model. Our goal in this section is to establish the following phase transition for magnetization.

Theorem 1. *Consider the Ising model on $\mathbb{L}^2 \cap [0, \sqrt{n}]^2$ on n vertices with no external field. Then*

1. $\mathbb{E}|\overline{x}| \geq \delta(\beta) > 0$ *for* $\beta > \beta_c$

2. $\lim_{n\to\infty} \mathbb{P}\left[|\overline{x}| \leq \delta\right] \to 1$ *for* $\beta < \beta_c$

The proof of this result will use Peierl's method of contours. We only will prove (1) in lecture for some β sufficiently close to 1. In particular, we will not show this holds for all β down to the critical temperature.

2 Contour Models

Recall that the dual lattice $G^* = (V^*, E^*)$ of $\mathbb{L}^2 \cap [0, \sqrt{n}]^2$ is the unit lattice on vertices in $(\mathbb{Z} + \frac{1}{2})^2$ such that there is exactly one dual edge crossing each edge of $\mathbb{L}^2 \cap [0, \sqrt{n}]^2$. The vertices of G^* lie in $[-1/2, \sqrt{n} + 1/2]^2$ excluding the four corners of this box and thus

$$|V^*| = \left(\sqrt{n} + 1\right)^2 - 4$$

A contour configuration $\mathscr{C} = \{C_1, C_2, \ldots, C_k\}$ is a set of directed paths in the dual lattice of $\mathbb{L}^2 \cap [0, \sqrt{n}]^2$ satisfying that:

1. Each path is closed or has two endpoints on the boundary of $[0, \sqrt{n}]^2$
2. No two paths cross
3. Two paths sharing a vertex bend to the right at that vertex
4. Pairs of contours such that one contains the other have opposite directions

The purpose of these conditions is to ensure that the contours partition $\mathbb{L}^2 \cap [0, \sqrt{n}]^2$ into regions each of which can be assigned either a $+$ or $-$ such that any path has $+$ on its left and $-$ on its right i.e. $+ \uparrow -$. These conditions ensure that assigning $+$'s and $-$'s to regions in this way is consistent. It is also important to ensure that the region outside all of the contours can be assigned a $+$ or $-$ in a consistent way. One way to do this is to assume the boundary also has a direction and to take paths p connecting two points on the boundary to be closed paths which are the union of p and a portion of the boundary consistent with the direction of p. A contour model is a distribution on contours given by

$$\mu_*(\mathscr{C}) = \frac{1}{Z_*} \prod_{c \in \mathscr{C}} e^{-2\beta|c|}$$

We now establish several properties of contour models that will be essential to the proof of Theorem 1.

Lemma 2. *Contour configurations are in bijection with Ising model configurations and this bijection maps the contour measure to the Ising measure.*

Proof. The bijection is given by assigning $+$ and $-$ to the regions of $\mathbb{L}^2 \cap [0, \sqrt{n}]^2$ defined by a contour configuration $\mathscr{C}$, as described above. The number of disagreeing interactions is

$$\#\text{disagreements} = \sum_{c \in \mathscr{C}} 2|c|$$

which implies that the Ising measure of the resulting spin configuration is proportional to $\prod_{c \in \mathscr{C}} e^{-2\beta|c|}$ and therefore given by $\mu_*(\mathscr{C})$. $\square$

Lemma 3. *The number of contours of length ℓ is*

$$N(\ell) \leq n \cdot 3^{\ell+1}$$

Proof. There are $|V^*|$ options for the starting vertex, at most 4 options for the first edge and at most 3 options for each remaining edge. Therefore

$$N(\ell) \leq |V^*| \cdot 4 \cdot 3^{\ell-1} \leq n \cdot 3^{\ell+1}$$

proving the lemma. $\square$

Lemma 4. $\mu_*(c \in \mathscr{C}) \leq e^{-2\beta|c|}$.

Proof. Given a contour configuration $\mathscr{C}$ with contour c, let $\mathscr{C} - c$ be the contour configuration formed by removing the contour c. In the context of Ising configurations, this corresponds to reversing the signs of all vertices inside the contour c. Now observe that

$$\begin{aligned}
\mu_*(\mathscr{C}) &= \sum_{\mathscr{C}: c \in \mathscr{C}} \mu_*(\mathscr{C}) \\
&= e^{-2\beta|c|} \sum_{\mathscr{C}: c \in \mathscr{C}} \frac{1}{Z^*} \prod_{c' \in \mathscr{C} - c} e^{-2\beta|c'|} \\
&= e^{-2\beta|c|} \sum_{\mathscr{C}: c \in \mathscr{C}} \mu_*(\mathscr{C} - c) \\
&\leq e^{-2\beta|c|}
\end{aligned}$$

because $\sum_{\mathscr{C}: c \in \mathscr{C}} \mu_*(\mathscr{C} - c)$ is a probability and at most 1. $\square$

3 Proof of Theorem 1 and Peierl's Argument

Let $P(+)$ be the set of all contour configurations x such that all $i \in V$ are inside a contour if $x_i = -1$. Similarly define $P(-)$ to be the set of all contour configurations such that $i \in V$ is inside some contour if $x_i = 1$. In other words $P(+)$ is the set of contour configurations with a "sea" of $+$. Now observe that

$$\begin{aligned}
\mathbb{E}|\overline{x}| &= \mathbb{E}\left[|\overline{x}| \cdot \mathbf{1}_{P(+)}\right] + \mathbb{E}\left[|\overline{x}| \cdot \mathbf{1}_{P(-)}\right] \\
&= 2 \cdot \mathbb{E}\left[|\overline{x}|\mathbf{1}_{P(+)}\right] \\
&\geq 2 \cdot \mathbb{E}\left[\overline{x} \cdot \mathbf{1}_{P(+)}\right] \\
&= 1 - \frac{2}{n} \cdot \mathbb{E}\left[\#(-) \cdot \mathbf{1}_{P(+)}\right] \\
&\geq 1 - \frac{2}{n} \cdot \mathbb{E}\left[\sum_{c \in \mathscr{C}} |c|^2\right] \\
&= 1 - \frac{2}{n} \cdot \sum_{c \in \mathscr{C}} |c|^2 \cdot \mu_*(c \in \mathscr{C}) \\
&\geq 1 - \frac{2}{n} \cdot \sum_{\ell \geq 2} N(\ell) e^{-2\beta\ell} \ell^2 \qquad \text{(Lemma 4)} \\
&\geq 1 - \frac{2}{n} \cdot \sum_{\ell \geq 2} n \cdot 3^{\ell+1} e^{-2\beta\ell} \ell^2 \qquad \text{(Lemma 3)} \\
&\geq \frac{1}{2}
\end{aligned}$$

for sufficiently large β. Similar ideas work to prove part (2) of Theorem 1.

4 Spatial and Temporal Mixing in the 2D Ising Model

The remainder of the lecture will be devoted to showing that spatial mixing and temporal mixing are equivalent for the 2D Ising model. Here, spatial mixing refers to exponential decay of correlations and temporal mixing refers to fast mixing of the Glauber dynamics.

Let ∂n denote the exterior vertex boundary of V_n where

$$V_n = \mathbb{L}^2 \cap \left[-\frac{\sqrt{n}}{2}, \frac{\sqrt{n}}{2}\right]$$

$$\partial n = \left\{v \in \mathbb{L}^2 : \text{dist}(v, V_n) = 1\right\}$$

Let $\pi^+(\cdot)$ and $\pi^-(\cdot)$ be the distributions of the Ising model on V_n conditioned on all $+$ and all $-$ boundary conditions, respectively, i.e.

$$\pi^+(\cdot) = \mathbb{P}(\cdot | x_{\partial n} = +)$$

$$\pi^-(\cdot) = \mathbb{P}(\cdot | x_{\partial n} = -)$$

Let ω denote the origin. Decay of correlations refers to the difference between $\pi^+(x_\omega = +)$ and $\pi^+(x_\omega = -)$ tending to zero as $n \to \infty$.

Theorem 5 (Temporal Mixing $\Rightarrow$ Spatial Mixing)**.** *Suppose that for any boundary conditions, the Glauber dynamics on the resulting 2D Ising model on V_n mixes in $O(n \log n)$ time. Then*

$$\left|\pi^+(x_\omega = +) - \pi^-(x_\omega = +)\right| \to 0$$

as $n \to \infty$.

Proof. The idea is to instantiate two instances of Glauber dynamics from a common initial state and the all $+$ and all $-$ boundary conditions, respectively. More precisely, let $X(t)$ and $Y(t)$ be Glauber dynamics for the two distributions $\pi^+(\cdot)$ and $\pi^-(\cdot)$ coupled as follows:

1. $X(0) = Y(0)$ is an arbitrary common initial configuration
2. Choose the same random vertex u for both chains at each step
3. If u is the random vertex chosen in a step of Glauber dynamics, make the same update to $X(t)_u$ and $Y(t)_u$ if $X(t)_{N(u)} = Y(t)_{N(u)}$
4. Make independent updates to $X(t)_u$ and $Y(t)_u$ if $X(t)_{N(u)} \neq Y(t)_{N(u)}$

Observe that by triangle inequality,

$$\begin{aligned} \left|\pi^+(x_\omega = +) - \pi^-(x_\omega = +)\right| &\leq \left|\pi^+(x_\omega = +) - \mathbb{P}[X(t)_\omega = +]\right| \\ &\quad + \left|\mathbb{P}[X(t)_\omega = +] - \mathbb{P}[Y(t)_\omega = +]\right| \\ &\quad + \left|\mathbb{P}[Y(t)_\omega = +] - \pi^-(x_\omega = +)\right| \\ &= \mathrm{I} + \mathrm{II} + \mathrm{III} \end{aligned}$$

Let $t_{\text{mix}} \leq cn \log n$ for both $X(t)$ and $Y(t)$. Then setting $t = cn \log^2 n$ implies

$$\mathrm{I}, \mathrm{III} \leq \frac{1}{n}$$

It now suffices to bound term II. We claim that $X_u(s) \neq Y_u(s)$ can happen only if $v \in N(u)$ was updated at a time $r \leq s$ or $X(0)_v \neq Y(0)_v$ for $v \in N(u)$. However, since $X(0) = Y(0)$ the second condition never occurs. Define a plausible path of disagreement to be a path $i_1, \ldots, i_\ell$ of vertices such that i_k is updated after i_{k-1}, $i_1 \in \partial n$ and $i_\ell = \omega$. It follows that for $X(t)_\omega \neq Y(t)_\omega$ to occur, there must be a plausible path of disagreement. Now,

$$\begin{aligned}
\text{II} &\leq \mathbb{P}\left[\exists \text{ plausible path of disagreement by time } t\right] \\
&\leq 4\sqrt{n} \sum_{k \geq \frac{\sqrt{n}}{2}} 3^k \binom{t}{k} \left(\frac{1}{n}\right)^k \\
&\leq 4\sqrt{n} \sum_{k \geq \frac{\sqrt{n}}{2}} \left(\frac{4et}{kn}\right)^k \\
&= 4\sqrt{n} \sum_{k \geq \frac{\sqrt{n}}{2}} \left(\frac{4e \log^2 n}{k}\right)^k \\
&\to 0 \text{ as } n \to \infty
\end{aligned}$$

The second inequality holds since any plausible path must have length $k \geq \sqrt{n}2$, there are $\binom{t}{k}$ options for the times at which the path grew, the probability of the path growing at these times exactly is n^{-k} and there are at most $4 \cdot 3^k \cdot \sqrt{n}$ plausible paths of disagreement. Note that there are at most $4 \cdot 3^k \cdot \sqrt{n}$ such paths because $|\partial n| = 4\sqrt{n}$ and there are at most 3 options at each step for the next edge. The third inequality follows from the bound $\binom{t}{k} \leq (te/k)^k$. Combining this with the bounds on I and III proves decay of correlations. $\square$

Theorem 6 (Spatial Mixing $\Rightarrow$ Temporal Mixing). *Suppose that*

$$|\pi^{i=+}(X_j = +) - \pi^{i=-}(X_j = +)| \leq e^{-\alpha d(i,j)}$$

for all $i, j \in V_n$ and all n where $\alpha > 0$ is independent of n. Then the Glauber dynamics on the 2D Ising model on V_n mixes in $O(n \log n)$ time for all boundary conditions.

The proof will be shown next lecture and will involve Block dynamics.

Definition 1 (Block Dynamics). *Fix some size L. Block dynamics is a Markov chain on 2D Ising models such that at each step, an $L \times L$ block is chosen uniformly at random and updated conditional on the boundary.*