3 problems, 46 points, 165 minutes.
1 sheet (2 pages) of notes allowed.

Good Luck!

PLEASE WRITE YOUR NAME ON EACH SHEET OF YOUR ANSWERS.

PLEASE WRITE THE SOLUTION OF EACH PROBLEM ON A SEPARATE SHEET.

Problem 1. (12 points)

Consider the following binary hypothesis testing problem. Suppose that under hypothesis $H = i \in \{0, 1\}$, we have

$$(Y_1, Y_2) = c_i + (Z_1, Z_2),$$

where $c_0 = (-1, -1)$ and $c_1 = (+1, +1)$. Define the functions T_1, T_2, T_3 as

$$T_1(Y_1, Y_2) = Y_1 - Y_2, \quad T_2(Y_1, Y_2) = Y_1 + Y_2, \quad T_3(Y_1, Y_2) = \operatorname{sign}(T_2).$$

(a) (3 pts) Suppose Z_1, Z_2 are i.i.d. $\mathcal{N}(0, 1)$. Is T_1 a sufficient statistic? Repeat for T_2 and T_3.

For the rest of the problem, suppose that Z_1, Z_2 are i.i.d. but Laplacian (i.e., each has probability density $f_Z(z) = \frac{1}{2}\exp(-|z|)$).

(b) (2 pts) What are the log likelihood ratios for the observed values $(y_1 = 4, y_2 = 0)$ and $(y_1 = 2, y_2 = 2)$?
Hint: Log likelihood ratio, $\mathrm{LLR}(y_1, y_2)$, is $\ln \frac{f_{Y_1,Y_2|H}(y_1,y_2|1)}{f_{Y_1,Y_2|H}(y_1,y_2|0)}$.

(c) (2 pts) Is T_2 a sufficient statistic? Justify your answer.
Hint: (b) might be useful.

(d) (3 pts) Show that when $\mathrm{LLR}(y_1, y_2) > 0$ we have $y_1 + y_2 > 0$, and when $\mathrm{LLR}(y_1, y_2) < 0$ we have $y_1 + y_2 < 0$.

(e) (2 pts) Suppose the hypotheses are equally likely. One person implements the MAP decision rule $\hat{H}_{\mathrm{MAP}}(Y_1, Y_2)$. Can someone who only observes T_2 implement a decision rule with the same error probability?

Problem 2. (16 points)

In a 3-ary hypothesis test with a priori equally likely hypotheses, when the true hypothesis $H = i \in \{0, 1, 2\}$, the observation $Y \in \mathbb{R}^n$ is given by

$$Y = c_i + Z,$$

with $c_i = x_i v$, where $x_0 = -2$, $x_1 = 0$, $x_2 = 2$ are scalars, $v = (1, 1/2, 1/3, \ldots, 1/n) \in \mathbb{R}^n$, and $Z = (Z_1, \ldots, Z_n)$ where $Z_1, \ldots, Z_n$ are i.i.d. $\mathcal{N}(0, 1)$.

(a) (3 pts) Let $T_n = Y_1 + \cdots + Y_n$. Consider decision rules based only on the value of T_n. What is the rule that minimizes the error probability? What is the error probability of this rule? [Let $H_n = \sum_{j=1}^{n} \frac{1}{j}$. Write your answer in terms of H_n and the Q-function.]

(b) (2 pts) For which values (if any) of n is T_n a sufficient statistic?

(c) (3 pts) As n gets large, what is the error probability of the decision rule using T_n? *Hint:* Facts that might be useful: $H_n = \sum_{j=1}^{n} j^{-1} \approx \ln(n)$ for large n, $\sum_{j=1}^{\infty} j^{-2} = \pi^2/6$, $\sum_{j=1}^{\infty} j^{-3} = 1.2020\ldots$, $\sum_{j=1}^{\infty} j^{-4} = \pi^4/90$. Other sums available upon request... And $\lim_{n\to\infty} \frac{\ln(n)}{n^a} = 0$ for any $a > 0$.

Consider $U_n = Y_1 + \frac{Y_2}{2} + \cdots + \frac{Y_n}{n}$.

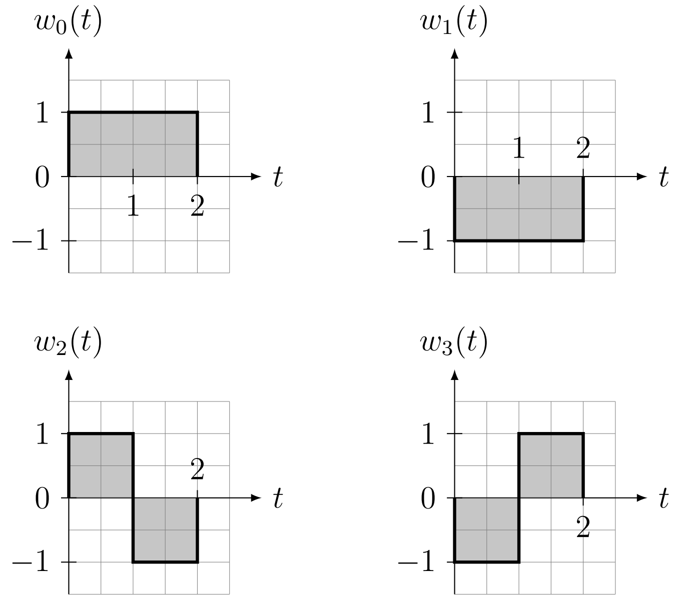
(d) (3 pts) Redo (a) with U_n replacing T_n.

(e) (2 pts) Redo (b) with U_n replacing T_n.

(f) (3 pts) Redo (c) with U_n replacing T_n.

Problem 3. (18 points)

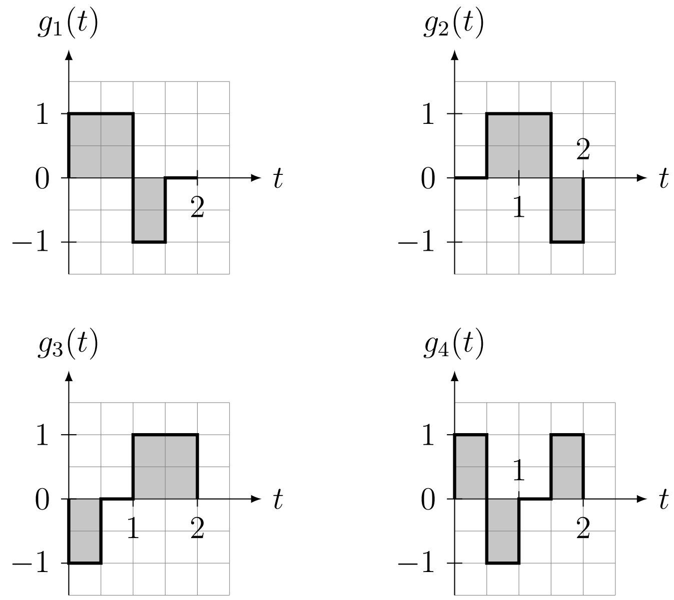
Suppose w_0, w_1, w_2 and w_3 are given in the following waveforms.



These waveforms are used to convey one of four messages over an AWGN channel with noise intensity σ^2. Assume that the four messages are equally likely.

(a) (3 pts) Draw the diagram of an optimal receiver architecture. If your architecture requires computing inner products, use matched filters to do so.

Suppose now we are given analog circuits which produce the inner products $U_\ell = \langle R, g_\ell \rangle$, $\ell = 1, 2, 3, 4$ of the received signal $R(t)$ with the waveforms g_1, g_2, g_3, g_4 given as follows:



Note that the g_ℓ's are cyclic shifts of each other in the interval $[0, 2]$.

(b) (3 pts) We are asked to design the best possible receiver *with no further analog circuits* (so: no other analog inner product computations, matched filters, etc.). Does this restriction penalize us (compared to what we are able to do in (a))? If yes, why? If no, why not?

Hint: First plot $4g_1(t) + 3g_2(t) + g_3(t) + 2g_4(t)$ and $4g_3(t) + 3g_4(t) + g_1(t) + 2g_2(t)$.

Suppose we replace the four waveforms $g_1, \dots, g_4$ above with only two waveforms $\tilde{g}_1(t) = \mathbb{1}_{[-1,1)}(t)$ and $\tilde{g}_2(t) = \mathbb{1}_{[1,3)}(t)$, with the corresponding inner products with $R(t)$ denoted by $\tilde{U}_1$ and $\tilde{U}_2$. Our receiver is now supposed to base its decision only on $(\tilde{U}_1, \tilde{U}_2)$.

(c) (3 pts) Draw the MAP decision regions for the transmitted message in the $(\tilde{u}_1, \tilde{u}_2)$ plane. Let $\tilde{p}_e(\sigma)$ denote the error probability of this receiver as a function of σ. Similarly, let $p_e(\sigma)$ denote the error probability of the receiver in (a). Find the ratio of $\tilde{p}_e(\sigma/\sqrt{2})$ and $p_e(\sigma)$.

For the rest of the problem we continue with the optimal receiver found in (a).

(d) (3 pts) Conditioned on $i = 0$ being the sent message, find the probability that the receiver in (a) decides $\hat{i} = 1$; repeat for $\hat{i} = 2$, and $\hat{i} = 3$.

Suppose that the data to be sent is composed of two bits b_1 and b_2, with (b_1, b_2) taking the values `00`, `01`, `10` and `11` with equal probability. We will assign these four possible data values to the messages 0, 1, 2, 3. To transmit a particular bit pair, we send the waveform corresponding to the respective message. This is summarized as follows:

$$(b_1, b_2) \longrightarrow i \longrightarrow w_i(t) \longrightarrow R(t) \longrightarrow \hat{i} \longrightarrow (\hat{b}_1, \hat{b}_2)$$

(e) (3 pts) Consider assigning the data bit values `00`, `01`, `10` and `11` to messages 0, 1, 2, 3 as `00` $\to$ 0, `01` $\to$ 1, `10` $\to$ 2, `11` $\to$ 3. Conditioned on $(b_1, b_2) =$ `00`, what is the expected number of incorrectly received bits $(\hat{b}_1, \hat{b}_2)$? Repeat the question for "`01` is sent", "`10` is sent" and "`11` is sent".

Hint: For the "repeat"s: you don't have to do any calculations.

(f) (3 pts) Consider assigning the two-bit values to messages as `00` $\to$ 0, `01` $\to$ 3, `10` $\to$ 2, `11` $\to$ 1. Redo (e).