

SOLUTION 1.

- (a) We have a binary hypothesis testing problem: The hypothesis  $H$  is the answer you will select, and your decision will be based on the observation of  $\hat{H}_L$  and  $\hat{H}_R$ . Let  $H$  take value 1 if answer 1 is chosen, and value 2 if answer 2 is chosen. In this case, we can write the MAP decision rule as follows:

$$\Pr\{H = 1 | \hat{H}_L = 1, \hat{H}_R = 2\} \underset{\hat{H}=2}{\overset{\hat{H}=1}{\geq}} \Pr\{H = 2 | \hat{H}_L = 1, \hat{H}_R = 2\}$$

From the problem setting we know the priors  $\Pr\{H = 1\}$  and  $\Pr\{H = 2\}$ ; we can also determine the conditional probabilities  $\Pr\{\hat{H}_L = 1 | H = 1\}$ ,  $\Pr\{\hat{H}_L = 1 | H = 2\}$ ,  $\Pr\{\hat{H}_R = 2 | H = 1\}$  and  $\Pr\{\hat{H}_R = 2 | H = 2\}$  (we have  $\Pr\{\hat{H}_L = 1 | H = 1\} = 0.9$  and  $\Pr\{\hat{H}_L = 1 | H = 2\} = 0.1$ ). Introducing these quantities and using the Bayes rule we can formulate the MAP decision rule as

$$\frac{\Pr\{\hat{H}_L = 1, \hat{H}_R = 2 | H = 1\} \Pr\{H = 1\}}{\Pr\{\hat{H}_L = 1, \hat{H}_R = 2\}} \underset{\hat{H}=2}{\overset{\hat{H}=1}{\geq}} \frac{\Pr\{\hat{H}_L = 1, \hat{H}_R = 2 | H = 2\} \Pr\{H = 2\}}{\Pr\{\hat{H}_L = 1, \hat{H}_R = 2\}}$$

Now, assuming that the event  $\{\hat{H}_L = 1\}$  is independent of the event  $\{\hat{H}_R = 2\}$  and simplifying the expression, we obtain

$$\Pr\{\hat{H}_L = 1 | H = 1\} \Pr\{\hat{H}_R = 2 | H = 1\} \Pr\{H = 1\} \underset{\hat{H}=2}{\overset{\hat{H}=1}{\geq}} \Pr\{\hat{H}_L = 1 | H = 2\} \Pr\{\hat{H}_R = 2 | H = 2\} \Pr\{H = 2\},$$

which is our final decision rule.

- (b) Evaluating the previous decision rule, we have

$$0.9 \times 0.3 \times 0.25 \underset{\hat{H}=2}{\overset{\hat{H}=1}{\geq}} 0.1 \times 0.7 \times 0.75,$$

which gives

$$0.0675 \underset{\hat{H}=2}{\overset{\hat{H}=1}{\geq}} 0.0525$$

This implies that the answer  $\hat{H}$  is equal to 1.

SOLUTION 2.

(a) We can write the MAP decision rule in the following way:

$$\frac{P_{Y|H}(y|1)}{P_{Y|H}(y|0)} \underset{\hat{H}=0}{\overset{\hat{H}=1}{\geq}} \frac{P_H(0)}{P_H(1)}$$

Plugging in, we find

$$\frac{\lambda_1^y e^{-\lambda_1}}{\lambda_0^y e^{-\lambda_0}} \underset{\hat{H}=0}{\overset{\hat{H}=1}{\geq}} \frac{p_0}{1-p_0},$$

and then

$$\left(\frac{\lambda_1}{\lambda_0}\right)^y \underset{\hat{H}=0}{\overset{\hat{H}=1}{\geq}} \frac{p_0}{1-p_0} e^{\lambda_1-\lambda_0}$$

Taking logarithms on both sides does not change the direction of the inequalities, therefore

$$y \log\left(\frac{\lambda_1}{\lambda_0}\right) \underset{\hat{H}=0}{\overset{\hat{H}=1}{\geq}} \log\left(\frac{p_0}{1-p_0} e^{\lambda_1-\lambda_0}\right)$$

Attention: the term  $\log(\lambda_1/\lambda_0)$  can be negative, and if it is, then dividing by it involves changing the direction of the inequality.

Suppose  $\lambda_1 > \lambda_0$ . Then,  $\log(\lambda_1/\lambda_0) > 0$ , and the decision rule becomes

$$y \underset{\hat{H}=0}{\overset{\hat{H}=1}{\geq}} \frac{\log\left(\frac{p_0}{1-p_0} e^{\lambda_1-\lambda_0}\right)}{\log\left(\frac{\lambda_1}{\lambda_0}\right)} \stackrel{\text{def}}{=} \theta$$

(b) We compute

$$\begin{aligned} P_e(0) &= \Pr\{Y > \theta | H = 0\} = \sum_{y=\lceil\theta\rceil}^{\infty} P_{Y|H}(y|0) \\ &= 1 - \sum_{y=0}^{\lfloor\theta\rfloor} \frac{\lambda_0^y}{y!} e^{-\lambda_0}, \end{aligned}$$

and by analogy

$$\begin{aligned} P_e(1) &= \Pr\{Y < \theta | H = 1\} = \sum_{y=0}^{\lfloor\theta\rfloor} P_{Y|H}(y|1) \\ &= \sum_{y=0}^{\lfloor\theta\rfloor} \frac{\lambda_1^y}{y!} e^{-\lambda_1} \end{aligned}$$

Thus, the probability of error becomes

$$P_e = p_0 \left(1 - \sum_{y=0}^{\lfloor\theta\rfloor} \frac{\lambda_0^y}{y!} e^{-\lambda_0}\right) + (1-p_0) \sum_{y=0}^{\lfloor\theta\rfloor} \frac{\lambda_1^y}{y!} e^{-\lambda_1}$$

Now, suppose that  $\lambda_1 < \lambda_0$ . Then,  $\log(\lambda_1/\lambda_0) < 0$ , and we have to swap the inequality sign, thus

$$y \underset{\substack{\hat{H}=0 \\ \geq \\ \hat{H}=1}}{\gtrless} \frac{\log\left(\frac{p_0}{1-p_0} e^{\lambda_1-\lambda_0}\right)}{\log\left(\frac{\lambda_1}{\lambda_0}\right)} \stackrel{\text{def}}{=} \theta$$

The rest of the analysis goes along the same lines, and finally, we obtain

$$P_e = p_0 \sum_{y=0}^{\lfloor \theta \rfloor} \frac{\lambda_0^y}{y!} e^{-\lambda_0} + (1-p_0) \left( 1 - \sum_{y=0}^{\lfloor \theta \rfloor} \frac{\lambda_1^y}{y!} e^{-\lambda_1} \right)$$

The case  $\lambda_0 = \lambda_1$  yields  $\log(\lambda_1/\lambda_0) = 0$ , so the decision rule becomes  $0 \underset{\substack{\hat{H}=1 \\ \geq \\ \hat{H}=0}}{\gtrless} \theta$ , regardless of  $y$ . Thus, we can exclude the case  $\lambda_0 = \lambda_1$  from our discussion.

(c) Here, we are in the case  $\lambda_1 > \lambda_0$ , and we find  $\theta \approx 4.54$ . We thus evaluate

$$P_e = \frac{1}{3} \left( 1 - \sum_{y=0}^4 \frac{2^y}{y!} e^{-2} \right) + \frac{2}{3} \sum_{y=0}^4 \left( \frac{10^y}{y!} e^{-10} \right) \approx 0.03705$$

(d) We find  $\theta \approx 7.5163$

$$P_e = \frac{1}{3} \left( 1 - \sum_{y=0}^7 \frac{2^y}{y!} e^{-2} \right) + \frac{2}{3} \sum_{y=0}^7 \left( \frac{20^y}{y!} e^{-20} \right) \approx 0.000885$$

The two Poisson distributions are much better separated than in (c); therefore, it becomes considerably easier to distinguish them based on one single observation  $y$ .

**SOLUTION 3.** We use the Fisher–Neyman factorization theorem.

(a) Since  $Y$  is an i.i.d. sequence,

$$\begin{aligned} P_{Y|H}(y|i) &= \prod_{k=1}^n P_{Y_k|H}(y_k|i) = \frac{\lambda_i^{\sum_{k=1}^n y_k}}{\prod_{k=1}^n (y_k)!} e^{-n\lambda_i} \\ &= \underbrace{e^{-n\lambda_i} \lambda_i^{n(\frac{1}{n} \sum_{k=1}^n y_k)}}_{g_i(T(y))} \underbrace{\frac{1}{\prod_{k=1}^n (y_k)!}}_{h(y)} \end{aligned}$$

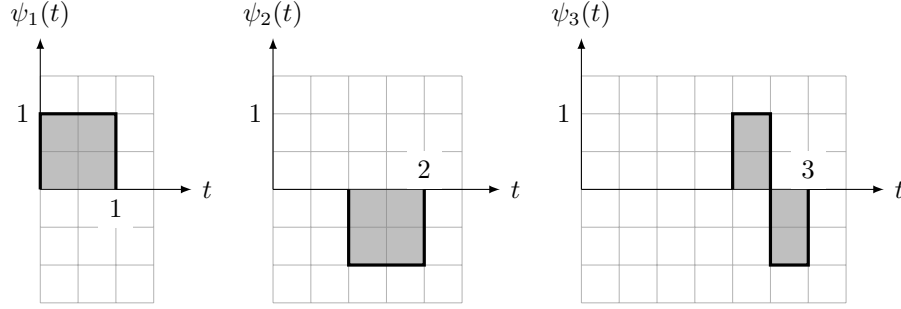
(b) Since  $Z_1, \dots, Z_n$  are i.i.d. additive noise samples,

$$\begin{aligned} f_{Y|H}(y|i) &= \prod_{k=1}^n f_{Z_k|H}(y_k - \theta_i) = \prod_{k=1}^n \lambda_i e^{-\lambda_i(y_k - \theta_i)} \mathbb{1}\{y_k \geq \theta_i\} \\ &= \underbrace{\lambda_i^n e^{n\lambda_i \theta_i} e^{-n\lambda_i(\frac{1}{n} \sum_{k=1}^n y_k)} \mathbb{1}\{\min\{y_1, \dots, y_n\} \geq \theta_i\}}_{g_i(T(y))} \end{aligned}$$

with  $h(y) = 1$ .

SOLUTION 4.

- (a) It is straightforward to check that  $w_0(t)$  has unit norm, i.e.,  $\|w_0(t)\| = 1$ , thus  $\psi_1(t) = w_0(t)$ . With  $\psi_1(t)$  we can reproduce the first portion of  $w_1(t)$  (for  $t$  between 0 and 1). With  $\psi_2(t)$  we need to be able to describe the remaining part of  $w_1(t)$ . Clearly  $\psi_2(t)$  is as illustrated below. With  $\psi_1(t)$  and  $\psi_2(t)$  we also describe the part of  $w_2(t)$  between  $t = 0$  and  $t = 2$ . Hence  $\psi_3(t)$  is selected as the unit-norm function that matches the part of  $w_2(t)$  between  $t = 2$  and  $t = 3$ . We immediately see that  $w_3(t)$  is also a linear combination of  $\psi_i(t)$ ,  $i = 1, 2, 3$ .



- (b) Using the basis  $\{\psi_1(t), \psi_2(t), \psi_3(t)\}$ , one can give the following representation for the waveforms  $w_i(t)$ ,  $i = 0, \dots, 3$ :

$$w_0 = (1, 0, 0)^T, w_1 = (-1, 1, 0)^T, w_2 = (1, 1, 1)^T, w_3 = (1, 1, -1)^T$$

SOLUTION 5.

- (a) The optimal solution is to pass  $R(t)$  through the matched filter  $w(T - t)$  and sample the result at  $t = T$  to get a sufficient statistic denoted by  $Y$ . (In this problem,  $T = 1$ .) Note that  $Y = S + N$ , where  $S$  and  $N$  are random variables denoting the signal and the noise components respectively. Under  $H = i$ ,  $Y \sim \mathcal{N}(\alpha_i, N_0/2)$ , where  $\alpha_0, \dots, \alpha_3$  are  $3c$ ,  $c$ ,  $-c$  and  $-3c$  respectively.

Let  $\hat{X}$  be the recovered signal value at the receiver. Based on the nearest neighbor decision rule, the receiver chooses the value of  $\hat{X}$  in the following fashion:

$$\hat{X} = \begin{cases} +3, & Y \in [2c, \infty) \\ +1, & Y \in [0, 2c) \\ -1, & Y \in [-2c, 0) \\ -3, & Y \in [-\infty, -2c) \end{cases} \quad (1)$$

- (b) The probability of error is given by

$$\begin{aligned} P_e &= \sum_{i=0}^3 \frac{1}{4} \Pr\{\text{error} | H = i\} \\ &= \frac{1}{4} \left[ Q\left(\frac{c}{\sqrt{N_0/2}}\right) + 2Q\left(\frac{c}{\sqrt{N_0/2}}\right) + 2Q\left(\frac{c}{\sqrt{N_0/2}}\right) + Q\left(\frac{c}{\sqrt{N_0/2}}\right) \right] \\ &= \frac{3}{2} Q\left(\frac{c}{\sqrt{N_0/2}}\right) \end{aligned}$$

- (c) In this case under  $H = i$ ,  $Y \sim \mathcal{N}(\alpha_i, N_0/2)$ , where  $\alpha_0, \dots, \alpha_3$  are  $\frac{9c}{4}, \frac{3c}{4}, \frac{-3c}{4}$  and  $\frac{-9c}{4}$  respectively. Using the decision rule in (1), the probability of error is given by

$$\begin{aligned}
P_e &= \sum_{i=0}^3 \frac{1}{4} \Pr\{\text{error}|H = i\} \\
&= \frac{1}{4} \left[ Q\left(\frac{c/4}{\sqrt{N_0/2}}\right) + Q\left(\frac{5c/4}{\sqrt{N_0/2}}\right) + Q\left(\frac{3c/4}{\sqrt{N_0/2}}\right) \right. \\
&\quad \left. + Q\left(\frac{5c/4}{\sqrt{N_0/2}}\right) + Q\left(\frac{3c/4}{\sqrt{N_0/2}}\right) + Q\left(\frac{c/4}{\sqrt{N_0/2}}\right) \right] \\
&= \frac{1}{2} \left[ Q\left(\frac{c/4}{\sqrt{N_0/2}}\right) + Q\left(\frac{3c/4}{\sqrt{N_0/2}}\right) + Q\left(\frac{5c/4}{\sqrt{N_0/2}}\right) \right]
\end{aligned}$$

- (d) The noise process  $N(t)$  is a stationary Gaussian random process. So the noise component  $N$  (which is the sample of match-filter output at time  $T$ ) is a Gaussian random variable with mean

$$\mathbb{E}[N] = \mathbb{E} \left[ \int_{-\infty}^{\infty} N(t)w(t)dt \right] = \mathbb{E} \left[ \int_0^1 N(t)dt \right] = 0$$

Because the process  $N(t)$  is stationary, without loss of generality we choose the boundaries of the integral to be 0 and  $T$  where in this problem  $T = 1$ .

Now, let us calculate the noise variance.

$$\begin{aligned}
\text{var}(N) &= \mathbb{E}[N^2] - \mathbb{E}[N]^2 = \mathbb{E}[N^2] \\
&= \mathbb{E} \left[ \int_{-\infty}^{\infty} N(t)w(t)dt \int_{-\infty}^{\infty} N(v)w(v)dv \right] \\
&= \mathbb{E} \left[ \int_0^1 N(t)dt \int_0^1 N(v)dv \right] \\
&= \mathbb{E} \left[ \int_0^1 \int_0^1 N(t)N(v)dt dv \right] \\
&= \int_0^1 \int_0^1 K_N(t-v)dt dv \\
&= \int_0^1 \int_0^1 \frac{1}{4\alpha} e^{-|t-v|/\alpha} dt dv \\
&= \frac{1}{2} (\alpha (e^{-1/\alpha} - 1) + 1)
\end{aligned}$$

Thus the new probability of error is given by

$$\begin{aligned}
P_e &= \sum_{i=0}^3 \frac{1}{4} \Pr\{\text{error}|H = i\} \\
&= \frac{1}{4} \left[ Q\left(\frac{c}{\sqrt{\text{var}(N)}}\right) + 2Q\left(\frac{c}{\sqrt{\text{var}(N)}}\right) + 2Q\left(\frac{c}{\sqrt{\text{var}(N)}}\right) + Q\left(\frac{c}{\sqrt{\text{var}(N)}}\right) \right] \\
&= \frac{3}{2} Q\left(\frac{c}{\sqrt{\frac{1}{2} (\alpha (e^{-1/\alpha} - 1) + 1)}}\right)
\end{aligned}$$