

ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

Handout 11

Principles of Digital Communications

Solutions to Problem Set 5

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SOLUTION 1.

- (a) At first look it may seem that the probability is uniformly distributed over the disk, but in the next part we will show that this is not true.
- (b) We know that R is uniformly distributed in $[0, 1]$ and Φ is uniformly distributed in $[0, 2\pi)$, so we have $f_R(r) = 1$ if $0 \leq r \leq 1$ and $f_\Phi(\phi) = \frac{1}{2\pi}$ if $0 \leq \phi < 2\pi$.

As these two random variables are independent, we have

$$f_{R,\Phi}(r, \phi) = \begin{cases} \frac{1}{2\pi} & 0 \leq r \leq 1 \text{ and } 0 \leq \phi < 2\pi \\ 0 & \text{otherwise.} \end{cases}$$

It can be easily shown that the Jacobian determinant is $\det J = r = \sqrt{x^2 + y^2}$. Therefore, the probability distribution in cartesian coordinates is

$$\begin{aligned} f_{X,Y}(x, y) &= \frac{1}{|\det J|} f_{R,\Phi}(r, \phi) \\ &= \begin{cases} \frac{1}{2\pi\sqrt{x^2+y^2}} & x^2 + y^2 \leq 1 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

- (c) We see that the probability distribution is not distributed uniformly. This makes sense because rings of equal width have the same probability but not the same area.

SOLUTION 2.

- (a) Let the two hypotheses be $H = 0$ and $H = 1$ when c_0 and c_1 are transmitted, respectively. The ML decision rule is

$$f_{Y_1 Y_2 | H}(y_1, y_2 | 1) \underset{\hat{H}=0}{\overset{\hat{H}=1}{\geq}} f_{Y_1 Y_2 | H}(y_1, y_2 | 0).$$

Because Z_1 and Z_2 are independent, we can write

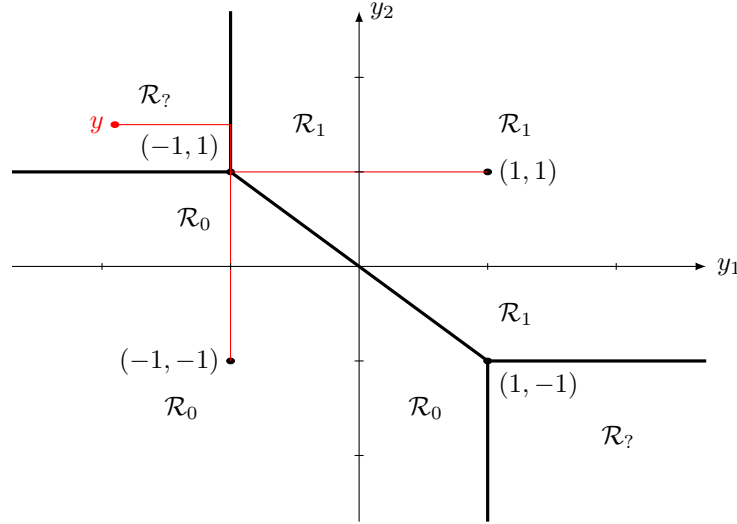
$$\frac{1}{2} e^{-|y_1-1|} \frac{1}{2} e^{-|y_2-1|} \underset{\hat{H}=0}{\overset{\hat{H}=1}{\geq}} \frac{1}{2} e^{-|y_1+1|} \frac{1}{2} e^{-|y_2+1|},$$

and, after taking the logarithm,

$$|y_1 + 1| + |y_2 + 1| \underset{\hat{H}=0}{\overset{\hat{H}=1}{\geq}} |y_1 - 1| + |y_2 - 1|.$$

- (b) Because the hypotheses are equally likely and Z_1 and Z_2 have the same distribution, the decision region for $\hat{H} = 0$ contains the points closer to $(-1, -1)$ and the decision region for $\hat{H} = 1$ contains the points closer to $(1, 1)$. For this problem, the distance between the points (y_{11}, y_{12}) and (y_{21}, y_{22}) is the Manhattan distance, $|y_{11} - y_{21}| + |y_{12} - y_{22}|$, and not the Euclidian distance.

Let us first consider the points above the line $y_2 = -y_1$ in the figure below. It is easy to notice that the points in the positive quadrant are closer to $(1, 1)$ than to $(-1, -1)$, therefore they belong to $\mathcal{R}_1$ ($\hat{H} = 1$). This is also true if $\{(y_1 \geq -y_2) \cap (y_2 \in (-1, 0))\}$, or if $\{(y_2 \geq -y_1) \cap (y_1 \in (-1, 0))\}$.



Similar reasoning can be applied to the points below the diagonal to determine $\mathcal{R}_0$.

The points for which $\{(y_1 \leq -1) \cap (y_2 \geq 1)\}$ or $\{(y_1 \geq 1) \cap (y_2 \leq -1)\}$ are equally distanced to $(-1, -1)$ and $(1, 1)$, therefore they can belong to either $\mathcal{R}_0$ or $\mathcal{R}_1$ with the same probability. This region is named $\mathcal{R}_?$.

- (c) The two hypotheses are equally probable for the region $\mathcal{R}_?$. Therefore, we can split this region in any way between the decision regions and have the same error probability. Because $\mathcal{R}_1$ is included in the region for which $y_2 > -y_1$ and $\mathcal{R}_0$ does not intersect the region for which $y_2 > -y_1$, the error probability is minimized by deciding $\hat{H} = 1$ if $(y_1 + y_2) > 0$.

- (d)

$$\begin{aligned}
 P_e(0) &= \Pr\{Y_1 + Y_2 > 0 | H = 0\} \\
 &= \Pr\{Z_1 + Z_2 - 2 > 0\} \\
 &= \int_2^\infty \frac{e^{-w}}{4} (1 + w) dw \\
 &= \frac{-e^{-w}}{4} (w + 2) \Big|_2^\infty = e^{-2}.
 \end{aligned}$$

By symmetry, and considering that the messages are equally likely, $P_e(0) = P_e(1) = P_e$.

SOLUTION 3.

- (a) The third component of c_i is zero for all i . Furthermore Z_1 , Z_2 and Z_3 are zero mean i.i.d. Gaussian random variables. Hence,

$$f_{Y|H}(y|i) = f_{Z_1}(y_1 - c_{i,1})f_{Z_2}(y_2 - c_{i,2})f_{Z_3}(y_3),$$

which is in the form $g_i(T(y))h(y)$ for $T(y) = (y_1, y_2)^\top$ and $h(y) = f_{Z_3}(y_3)$. Hence, by the Fisher–Neyman factorization theorem, $T(Y) = (Y_1, Y_2)^\top$ is a sufficient statistic.

- (b) We have $Y_3 = Z_3 = Z_2$. By observing Y_3 , we can remove the noise in the second component of Y . Specifically, we have $c_{i,2} = Y_2 - Y_3$. If the second component is different for each hypothesis, then the receiver can make an error-free decision which is not possible using only $(Y_1, Y_2)^\top$ (see the next question for more on this). We can see that Y_3 contains very useful information and can't be discarded. Therefore, $(Y_1, Y_2)^\top$ is not a sufficient statistic.

- (c) If we have only $(Y_1, Y_2)^\top$ then the hypothesis testing problem will be

$$H = i : (Y_1, Y_2) = (c_{i,1}, c_{i,2}) + (Z_1, Z_2) \quad i = \{0, 1\}$$

Using the fact that $c_0 = (1, 0, 0)^\top$ and $c_1 = (0, 1, 0)^\top$, the ML test becomes

$$y_1 - y_2 \underset{\hat{H}=1}{\overset{\hat{H}=0}{\gtrless}} 0$$

Under $H = 0$, $Y_1 - Y_2$ is a Gaussian random variable with mean 1 and variance $2\sigma^2$, and so $P_e(0) = Q(\frac{1}{\sqrt{2}\sigma})$. By symmetry $P_e(1) = Q(\frac{1}{\sqrt{2}\sigma})$, and so the error probability will be $P_e = \frac{1}{2}(P_e(0) + P_e(1)) = Q(\frac{1}{\sqrt{2}\sigma})$.

Now assume that we have access to Y_1 , Y_2 and Y_3 . Y_3 contains $Z_3 = Z_2$ under both hypotheses. Hence, $Y_2 - Y_3 = c_{i,2} + Z_2 - Z_3 = c_{i,2}$. This shows that at the receiver we can observe the second component of c_i without noise. As the second component is different under both hypotheses, we can make an error-free decision about H and the decision rule will be:

$$\hat{H} = \begin{cases} 0 & y_2 - y_3 = 0 \\ 1 & y_2 - y_3 = 1 \end{cases}$$

Clearly this decision rule minimizes the error probability. This shows once again that $(Y_1, Y_2)^\top$ can't be a sufficient statistic.

SOLUTION 4.

- (a) We use the Gram-Schmidt procedure:

- 1) The first step is to normalize the function $\beta_0(t)$, i.e. the first function of the basis that we are looking for is

$$\begin{aligned} \psi_0(t) &= \frac{\beta_0(t)}{\|\beta_0(t)\|} = \frac{\beta_0(t)}{\sqrt{\int \beta_0(t)^2 dt}} \\ &= \frac{\beta_0(t)}{\sqrt{\int_0^1 4t^2 dt}} = \frac{\sqrt{3}}{2}\beta_0(t) = \begin{cases} 0 & \text{if } t < 0 \\ \sqrt{3}t & \text{if } 0 \leq t \leq 1 \\ 0 & \text{if } t > 1 \end{cases} \end{aligned}$$

- 2) Next, we subtract from $\beta_1(t)$ the components that are in the span of the currently established part of the basis, i.e. in the span of $\{\psi_0(t)\}$. This can be achieved by projecting $\beta_1(t)$ onto $\psi_0(t)$ and then subtracting this projection from $\beta_1(t)$, i.e.

$$\begin{aligned}\alpha_1(t) &= \beta_1(t) - \langle \beta_1(t), \psi_0(t) \rangle \psi_0(t) = \beta_1(t) - \left(\int \beta_1(t) \psi_0(t) dt \right) \psi_0(t) \\ &= \beta_1(t) - \left(\frac{\sqrt{3}}{2} \right) \left(\frac{4}{3} \right) \psi_0(t) \\ &= \beta_1(t) - \frac{2}{\sqrt{3}} \psi_0(t) \\ &= \beta_1(t) - \beta_0(t).\end{aligned}$$

From this, we find the second basis element as

$$\psi_1(t) = \frac{\alpha_1(t)}{\|\alpha_1(t)\|} = \begin{cases} 0 & \text{if } t < 1 \\ -\sqrt{3}(t-2) & \text{if } 1 \leq t \leq 2 \\ 0 & \text{if } t > 2 \end{cases}$$

- 3) Again, we subtract from $\beta_2(t)$ the components that are in the span of the currently established part of the basis, i.e. in the span of $\{\psi_0(t), \psi_1(t)\}$. This can be achieved by projecting $\beta_2(t)$ onto $\psi_0(t)$ and $\psi_1(t)$ and then subtracting both these projections from $\beta_2(t)$. For this step, it is *essential* that the basis elements $\{\psi_0(t), \psi_1(t)\}$ be orthonormal. Continuing the derivation, we obtain

$$\begin{aligned}\alpha_2(t) &= \beta_2(t) - \langle \beta_2(t), \psi_0(t) \rangle \psi_0(t) - \langle \beta_2(t), \psi_1(t) \rangle \psi_1(t) \\ &= \beta_2(t) - \left(\int \beta_2(t) \psi_0(t) dt \right) \psi_0(t) - \left(\int \beta_2(t) \psi_1(t) dt \right) \psi_1(t) \\ &= \beta_2(t) - 0 + \alpha_1(t) \\ &= \beta_2(t) - \beta_0(t) + \beta_1(t),\end{aligned}$$

and from this, we find the third basis element as

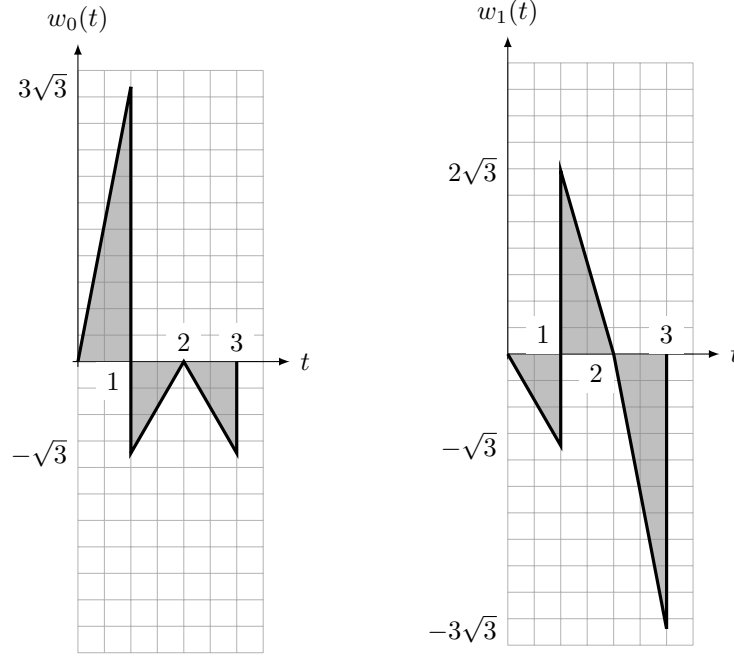
$$\psi_2(t) = \frac{\alpha_2(t)}{\|\alpha_2(t)\|} = \begin{cases} 0 & \text{if } t < 2 \\ -\sqrt{3}(t-2) & \text{if } 2 \leq t \leq 3 \\ 0 & \text{if } t > 3 \end{cases}$$

- (b) By definition we can write $w_0(t)$ and $w_1(t)$ as follows

$$w_0(t) = 3\psi_0(t) - \psi_1(t) + \psi_2(t) = \begin{cases} 3\sqrt{3}t & \text{if } 0 \leq t < 1 \\ \sqrt{3}(t-2) & \text{if } 1 < t < 2 \\ -\sqrt{3}(t-2) & \text{if } 2 < t \leq 3 \end{cases}$$

and

$$w_1(t) = -\psi_0(t) + 2\psi_1(t) + 3\psi_2(t) = \begin{cases} -\sqrt{3}t & \text{if } 0 \leq t < 1 \\ -2\sqrt{3}(t-2) & \text{if } 1 < t < 2 \\ -3\sqrt{3}(t-2) & \text{if } 2 < t \leq 3 \end{cases}$$



(c)

$$\langle c_0, c_1 \rangle = -3 \cdot 1 - 1 \cdot 2 + 1 \cdot 3 = -2.$$

We know that $w_0(t)$ and $w_1(t)$ are both real, thus

$$\begin{aligned} \langle w_0(t), w_1(t) \rangle &= \int w_0(t)w_1(t) dt = \int_0^1 -9t^2 dt + \int_1^2 -6(t-2)^2 dt + \int_2^3 9(t-2)^2 dt \\ &= - \int_1^2 6(t-2)^2 dt = -2. \end{aligned}$$

We see that the inner products are equal as expected.

(d)

$$\begin{aligned} \|c_0\| &= \sqrt{\langle c_0, c_0 \rangle} = \sqrt{11}, \\ \|w_0\|^2 &= \int |w_0(t)|^2 dt = \int_0^1 27t^2 dt + \int_1^3 3(t-2)^2 dt = 9 + 2 = 11. \end{aligned}$$

We see that the norms are also equal.

SOLUTION 5.

(a)

$$\|g_i\| = \sqrt{T}, \quad i = 1, 2, 3.$$

(b) Z_1 and Z_2 are independent since g_1 and g_2 are orthogonal. Hence Z is a Gaussian random vector $\sim \mathcal{N}(0, \sigma^2 I_2)$, where $\sigma^2 = \frac{N_0}{2}T$.

(c)

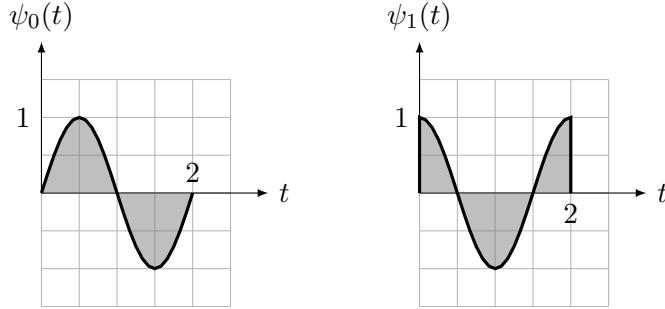
$$\begin{aligned} P_a &= \Pr\{Z_1 \in [1, 2] \cap Z_2 \in [1, 2]\} = \Pr\{Z_1 \in [1, 2]\} \Pr\{Z_2 \in [1, 2]\} \\ &= \left[Q\left(\frac{1}{\sigma}\right) - Q\left(\frac{2}{\sigma}\right) \right]^2, \end{aligned}$$

where $\sigma^2 = \frac{N_0}{2}T$.

- (d) $P_b = P_a$, since one obtains the square (b) from the square (a) via a rotation.
- (e) $Z_3 = -Z_1$. $U = Z_1(1, -1)^T$, and thus U can never be in (a), hence $Q_a = 0$.
- (f) U is in square (c) if and only if $Z_1 \in [1, 2]$. Hence $Q_c = Q\left(\frac{1}{\sigma}\right) - Q\left(\frac{2}{\sigma}\right)$, where $\sigma^2 = \frac{N_0}{2}T$.

SOLUTION 6.

- (a) An orthonormal basis for the signal space spanned by the waveforms is¹:



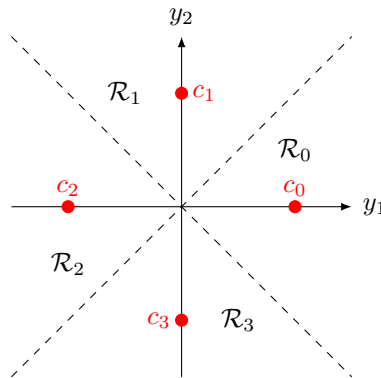
- (b) The codewords representing the waveforms are

$$\begin{aligned} c_0 &= (\sqrt{\mathcal{E}}, 0) \\ c_1 &= (0, \sqrt{\mathcal{E}}) \\ c_2 &= (-\sqrt{\mathcal{E}}, 0) \\ c_3 &= (0, -\sqrt{\mathcal{E}}) \end{aligned}$$

- (c) As we have seen in the lecture, if $R(t)$ is the noisy received waveform, $(Y_0, Y_1) = (\langle R, \psi_0 \rangle, \langle R, \psi_1 \rangle)$ is a sufficient statistic for decision. Hence, we have the following hypothesis testing problem: Under $H = i$, $i = 0, 1, 2, 3$,

$$Y_i = c_i + Z,$$

where $Z \sim \mathcal{N}(0, \frac{N_0}{2}I_2)$. One can check that c_i , $i = 0, 1, 2, 3$ represent the QPSK codewords, and the decision regions for the ML receiver will be as follows:



¹this can be obtained using the Gram-Schmidt procedure or simply by looking at the waveforms.

The distance between two adjacent codewords (say c_0 and c_1) is $d = \sqrt{2\mathcal{E}}$ and the error probability of the receiver is

$$\begin{aligned}
 P_e &= 2Q\left(\frac{d}{2\sigma}\right) - Q^2\left(\frac{d}{2\sigma}\right) \\
 &= 2Q\left(\frac{\sqrt{2\mathcal{E}}}{2\sqrt{N_0/2}}\right) - Q^2\left(\frac{\sqrt{2\mathcal{E}}}{2\sqrt{N_0/2}}\right) \\
 &= 2Q\left(\sqrt{\frac{\mathcal{E}}{N_0}}\right) - Q^2\left(\sqrt{\frac{\mathcal{E}}{N_0}}\right).
 \end{aligned}$$