ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
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Handout 2 Principles of Digital Communications
Problem Set 1 Feb. 21, 2024

PROBLEM 1. Assume that X_1 and X_2 are independent random variables that are uniformly distributed in the interval $[0, 1]$. Compute the probability of the following events.

(a) $0 \leq X_1 - X_2 \leq \frac{1}{3}$.

(b) $X_1^3 \leq X_2 \leq X_1^2$.

(c) $X_2 - X_1 = \frac{1}{2}$.

(d) $(X_1 - \frac{1}{2})^2 + (X_2 - \frac{1}{2})^2 \leq (\frac{1}{2})^2$.

(e) Given that $X_1 \geq \frac{1}{4}$, compute the probability that $(X_1 - \frac{1}{2})^2 + (X_2 - \frac{1}{2})^2 \leq (\frac{1}{2})^2$.

Hint: For each event, identify the corresponding region inside the unit square.

PROBLEM 2. Find the following probabilities.

(a) A box contains m white and n black balls. Suppose k balls are drawn. Find the probability of drawing at least one white ball.

(b) We have two coins; the first is fair and the second is two-headed. We pick one of the coins at random, toss it twice, and obtain heads both times. Find the probability that the coin is fair.

PROBLEM 3. Assume X and Y are random variables with joint density function

$$f_{X,Y}(x, y) = \begin{cases} A, & 0 \leq x < y \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Are X and Y independent?

(b) Compute the value of A.

(c) Find the density function of Y. Do this first by arguing geometrically, then compute it analytically.

(d) Find $\mathbb{E}[X|Y = y]$. *Hint:* Argue geometrically.

(e) The $\mathbb{E}[X|Y = y]$ found in (d) is a function of y, call it $f(y)$. Find $\mathbb{E}[f(Y)]$. This is $\mathbb{E}\big[\mathbb{E}[X|Y]\big]$.

(f) Find $\mathbb{E}[X]$ from the definition. Verify that $\mathbb{E}[X]$ is equal to $\mathbb{E}[\mathbb{E}[X|Y]]$ computed in (e). Is this a coincidence?

PROBLEM 4. Let Z_1 and Z_2 be i.i.d. zero-mean Gaussian random variables, i.e., the pdf of Z_i, $i = 1, 2$ is

$$f_Z(z) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{z^2}{2\sigma^2}}$$

for some $\sigma > 0$. Define

$$X := \frac{Z_1}{\sqrt{Z_1^2 + Z_2^2}} \qquad \text{and} \qquad Y := \frac{Z_2}{\sqrt{Z_1^2 + Z_2^2}}.$$

Prove that (X, Y) is a uniformly chosen point on the unit circle.

Problem 5.

(a) Let X and Y be two continuous real-valued random variables with joint probability density function $f_{X,Y}$. Show that if X and Y are independent, they are also *uncorrelated.*

(b) Consider two independent and uniformly distributed random variables $U \in \{0, 1\}$ and $V \in \{0, 1\}$. Assume that X and Y are defined as follows: $X = U + V$ and $Y = |U - V|$. Are X and Y independent? Compute the covariance of X and Y. What do you conclude?

Problem 6. Assume you pick a point on the surface of the unit sphere (i.e. the sphere centered at the origin with radius 1) uniformly at random and (X, Y, Z) denotes its coordinates (in 3D space). Compute $\mathbb{E}[X^2]$.

Problem 7. Assume the random variable X has an exponential distribution given by $f_X(x) = e^{-x}$ when $x \geq 0$. Similarly, $\hat{X}$ is exponentially distributed with $f_{\hat{X}}(\hat{x}) = 2e^{-2\hat{x}}$ for $\hat{x} \geq 0$.

(a) For what values of x do we have $f_X(x) \leq f_{\hat{X}}(x)$?

(b) Calculate $\mathbb{P}\left(f_X(X) \leq f_{\hat{X}}(X)\right)$.

(c) Calculate $\mathbb{P}\left(f_X(\hat{X}) \geq f_{\hat{X}}(\hat{X})\right)$.