

PROBLEM 1. Random variables  $X$  and  $Y$  are correlated Gaussian variables:

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim \mathcal{N}_2 \left( \begin{pmatrix} 0 \\ 0 \end{pmatrix} : K = \begin{bmatrix} \sigma_x^2 & \rho\sigma_x\sigma_y \\ \rho\sigma_x\sigma_y & \sigma_y^2 \end{bmatrix} \right).$$

Find  $I(X; Y)$ .

PROBLEM 2. Consider an additive noise channel with input  $x \in \mathbb{R}$ , and output

$$Y = x + Z$$

where  $Z$  is a real random variable independent of the input  $x$ , has zero mean and variance equal to  $\sigma^2$ .

In this problem we prove in a different way from the lecture that the Gaussian channel has the smallest capacity among all additive noise channels of a given noise variance. Let  $\mathcal{N}_{\sigma^2}$  denote the Gaussian density with zero mean and variance  $\sigma^2$ .

(a) Denote the input probability density by  $p_X$ . Verify that

$$I(X; Y) = \iint p_X(x)p_Z(y-x) \ln \frac{p_Z(y-x)}{p_Y(y)} dx dy \quad \text{nats.}$$

where  $p_Y$  is the density of the output when the input has density  $p_X$ .

(b) Now set  $p_X = \mathcal{N}_P$ . Verify that

$$\frac{1}{2} \ln(1 + P/\sigma^2) = \iint p_X(x)p_Z(y-x) \ln \frac{\mathcal{N}_{\sigma^2}(y-x)}{\mathcal{N}_{P+\sigma^2}(y)} dx dy.$$

(c) Still with  $p_X = \mathcal{N}_P$ , show that

$$\frac{1}{2} \ln(1 + P/\sigma^2) - I(X; Y) \leq 0.$$

[Hint: use (a) and (b) and  $\ln t \leq t - 1$ .]

(d) Show that an additive noise channel with noise variance  $\sigma^2$  and input power  $P$  has capacity at least  $\frac{1}{2} \log_2(1 + P/\sigma^2)$  bits per channel use. Conclude that the Gaussian channel has the smallest capacity among all additive noise channels of a given noise variance.

PROBLEM 3. A discrete memoryless channel has three input symbols:  $\{-1, 0, 1\}$ , and two output symbols:  $\{1, -1\}$ . The transition probabilities are

$$p(-1|-1) = p(1|1) = 1, \quad p(1|0) = p(-1|0) = 0.5.$$

Find the capacity of this channel with cost constraint  $\beta$ , if the cost function is  $b(x) = x^2$ .

PROBLEM 4. Consider a vector Gaussian channel described as follows:

$$\begin{aligned} Y_1 &= x + Z_1 \\ Y_2 &= Z_2 \end{aligned}$$

where  $x$  is the input to the channel constrained in power to  $P$ ;  $Z_1$  and  $Z_2$  are jointly Gaussian random variables with  $E[Z_1] = E[Z_2] = 0$ ,  $E[Z_1^2] = E[Z_2^2] = \sigma^2$  and  $E[Z_1 Z_2] = \rho\sigma^2$ , with  $\rho \in [-1, 1]$ , and independent of the channel input.

- (a) Consider a receiver that discards  $Y_2$  and decodes the message based only on  $Y_1$ . What rates are achievable with such a receiver?
- (b) Consider a receiver that forms  $Y = Y_1 - \rho Y_2$ , and decodes the message based only on  $Y$ . What rates are achievable with such a receiver?
- (c) Find the capacity of the channel and compare it to the part (b).

PROBLEM 5. Suppose  $(X_1, \dots, X_n, Y_1, \dots, Y_m)$  is an  $n + m$  dimensional Gaussian random vector with covariance matrix  $K$ . Partition the  $(n + m) \times (n + m)$  matrix  $K$  as

$$K = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}$$

where  $K_{11}$  is  $n \times n$  and  $K_{22}$  is  $m \times m$ .

- (a) Express  $h(X_1, \dots, X_n)$ ,  $h(Y_1, \dots, Y_m)$  and  $h(X_1, \dots, X_n, Y_1, \dots, Y_m)$  in terms of the matrices above.
- (b) Show if the matrix  $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$  is positive definite, then  $\det(A) \leq \det(A_{11}) \det(A_{22})$ . [Hint: for any positive definite matrix  $A$ ,  $f(x) = \det(2\pi A)^{-1/2} \exp(-\frac{1}{2}x^T A^{-1}x)$  is a probability density.]