ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

Handout 19 — Information Theory and Coding

Homework 8 — Nov. 13, 2023

PROBLEM 1. Show that a cascade of n identical binary symmetric channels,

$$X_0 \to \boxed{\text{BSC \#1}} \to X_1 \to \cdots \to X_{n-1} \to \boxed{\text{BSC \#n}} \to X_n$$

each with raw error probability p, is equivalent to a single BSC with error probability $\frac{1}{2}\big(1-(1-2p)^n\big)$ and hence that $\lim_{n\to\infty} I(X_0;X_n)=0$ if $p\neq 0,1$. Thus, if no processing is allowed at the intermediate terminals, the capacity of the cascade tends to zero.

PROBLEM 2. Consider a memoryless channel with transition probability matrix $P_{Y|X}(y|x)$, with $x\in\mathcal{X}$ and $y\in\mathcal{Y}$. For a distribution Q over $\mathcal{X}$, let $I(Q)$ denote the mutual information between the input and the output of the channel when the input distribution is Q. Show that for any two distributions Q and Q' over $\mathcal{X}$,

(a)
$$I(Q') \le \sum_{x\in\mathcal{X}} Q'(x) \sum_{y\in\mathcal{Y}} P_{Y|X}(y|x)\log\left(\frac{P_{Y|X}(y|x)}{\sum_{x'\in\mathcal{X}} P_{Y|X}(y|x')Q(x')}\right)$$

(b)
$$C \le \max_x \sum_{y\in\mathcal{Y}} P_{Y|X}(y|x)\log\left(\frac{P_{Y|X}(y|x)}{\sum_{x'\in\mathcal{X}} P_{Y|X}(y|x')Q(x')}\right)$$

where C is the capacity of the channel. Notice that this upper bound to the capacity is independent of the maximizing distribution.

PROBLEM 3.

(a) Show that $I(U;V)\ge I(U;V|T)$ if T, U, V form a Markov chain, i.e., conditional on U, the random variables T and V are independent.

Fix a conditional probability distribution $p(y|x)$, and suppose $p_1(x)$ and $p_2(x)$ are two probability distributions on $\mathcal{X}$.

For $k\in\{1,2\}$, let I_k denote the mutual information between X and Y when the distribution of X is $p_k(\cdot)$.

For $0\le\lambda\le 1$, let W be a random variable, taking values in $\{1,2\}$, with

$$\Pr(W=1)=\lambda,\quad \Pr(W=2)=1-\lambda.$$

Define
$$p_{W,X,Y}(w,x,y)=\begin{cases}\lambda p_1(x)p(y|x) & \text{if } w=1\\ (1-\lambda)p_2(x)p(y|x) & \text{if } w=2.\end{cases}$$

(b) Express $I(X;Y|W)$ in terms of I_1, I_2 and λ.

(c) Express $p(x)$ in terms of $p_1(x)$, $p_2(x)$ and λ.

(d) Using (a), (b) and (c) show that, for every fixed conditional distribution $p_{Y|X}$, the mutual information $I(X;Y)$ is a concave $\cap$ function of p_X.

PROBLEM 4. Suppose Z is uniformly distributed on $[-1,1]$, and X is a random variable, independent of Z, constrained to take values in $[-1,1]$. What distribution for X maximizes the entropy of $X+Z$? What distribution of X maximizes the entropy of XZ?

PROBLEM 5. Let P_1 and P_2 be two channels of input alphabet $\mathcal{X}_1$ and $\mathcal{X}_2$ and of output alphabet $\mathcal{Y}_1$ and $\mathcal{Y}_2$ respectively. Consider a communication scheme where the transmitter chooses the channel (P_1 or P_2) to be used and where the receiver knows which channel were used. This scheme can be formalized by the channel P of input alphabet $\mathcal{X} = (\mathcal{X}_1 \times \{1\}) \cup (\mathcal{X}_2 \times \{2\})$ and of output alphabet $\mathcal{Y} = (\mathcal{Y}_1 \times \{1\}) \cup (\mathcal{Y}_2 \times \{2\})$, which is defined as follows:

$$P(y,k'|x,k) = \begin{cases} P_k(y|x) & \text{if } k' = k, \\ 0 & \text{otherwise.} \end{cases}$$

Let $X = (X_k, K)$ be a random variable in $\mathcal{X}$ which will be the input distribution to the channel P, and let $Y = (Y_k, K) \in \mathcal{Y}$ be the output distribution. Define X_1 as being the random variable in $\mathcal{X}_1$ obtained by conditioning X_k on $K = 1$. Similarly define X_2, Y_1 and Y_2. Let α be the probability that $K = 1$.

(a) Show that $I(X;Y) = h_2(\alpha) + \alpha I(X_1;Y_1) + (1-\alpha) I(X_2;Y_2)$.

(b) What is the input distribution X that achieves the capacity of P?

(c) Show that the capacity C of P satisfies $2^C = 2^{C_1} + 2^{C_2}$, where C_1 and C_2 are the capacities of P_1 and P_2 respectively.

PROBLEM 6. Suppose X and Y are independent geometric random variables. That is, $p_X(k) = (1-p)^{k-1}p$ and $p_Y(k) = (1-q)^{k-1}q, \quad \forall k \in \{1,2,\ldots\}$.

(a) Find $H(X,Y)$.

(b) Find $H(2X+Y, X-2Y)$

Now consider two independent exponential random variables X and Y. That is, $p_X(t) = \lambda_X e^{-\lambda_X t}$ and $p_Y(t) = \lambda_Y e^{-\lambda_Y t}, \quad \forall t \in [0,\infty)$.

(c) Find $h(X,Y)$.

(d) Find $h(2X+Y, X-2Y)$