Problem 1. Since $L$ is linear, we know that

$$L(\lambda x) = \lambda L(x)$$

for any $\lambda \in \mathbb{R}$. Similarly, $g$ is concave so it must satisfy the following by definition.

$$g(\lambda x_1 + (1 - \lambda)x_2) \geq \lambda g(x_1) + (1 - \lambda)g(x_2)$$

for any $\lambda \in [0,1]$. Combining these two statements, the following steps show that $f$ is concave.

$$f(\lambda x_1 + (1 - \lambda)x_2) = g(L(\lambda x_1 + (1 - \lambda)x_2))$$
$$= g(\lambda L(x_1) + (1 - \lambda)L(x_2))$$
$$\geq \lambda g(L(x_1)) + (1 - \lambda)g(L(x_2))$$
$$= \lambda f(x_1) + (1 - \lambda)f(x_2)$$

where (1) uses the linearity property of $L$ and (2) uses the concavity property of $g$.

Problem 2.

(a) Let $s(m) = 0 + 1 + \cdots + (m - 1) = m(m - 1)/2$. Suppose we have a string of length $n = s(m)$. Then, we can certainly parse it into $m$ words of lengths $0, 1, \ldots, (m - 1)$, and since these words have different lengths, we are guaranteed to have a distinct parsing. Since a parsing with the maximal number of distinct words will have at least as many words as this particular parsing, we conclude that whenever $n = m(m - 1)/2$, $c \geq m$ (and for $n > m(m - 1)/2$ we can parse the first $m(m - 1)/2$ letters to $m$, as we just described, and append the remaining letters to the last word to have a parsing into $m$ distinct words).

(b) An all zero string of length $s(m)$ can be parsed into at most $m$ words: in this case distinct words must have distinct lengths and the bound is met with equality.

(c) Now, given $n$, we can find $m$ such that $s(m - 1) \leq n < s(m)$. A string with $n$ letters can be parsed into $m - 1$ distinct words by parsing its initial segment of $s(m - 1)$ letters with the above procedure, and concatenating the leftover letters to the last word. Thus, if a string can be parsed into $m - 1$ distinct words, then $n < s(m)$, and in particular, $n < s(c + 1) = c(c + 1)/2$. From above, it is clear that no sequence will meet the bound with equality.

Problem 3. Observe that $H(Y) - H(Y|X) = I(X; Y) = I(X; Z) = H(Z) - H(Z|X)$.

(a) Consider a channel with binary input alphabet $\mathcal{X} = \{0, 1\}$ with $X$ uniformly distributed over $\mathcal{X}$, output alphabet $\mathcal{Y} = \{0, 1, 2, 3\}$, and probability law

$$P_{Y|X}(y|x) = \begin{cases} \frac{1}{2}, & \text{if } x = 0 \text{ and } y = 0 \\ \frac{1}{2}, & \text{if } x = 0 \text{ and } y = 1 \\ \frac{1}{2}, & \text{if } x = 1 \text{ and } y = 2 \\ \frac{1}{2}, & \text{if } x = 1 \text{ and } y = 3 \\ 0, & \text{otherwise.} \end{cases}$$
It is easy to verify $H(Y|X) = 1$. Since $Y$ takes any value in $\mathcal{Y}$ with equal probability, its entropy is $H(Y) = 2$. Therefore $I(X;Y) = 1$. Define the processor output to be in alphabet $\mathcal{Z}$ and construct a deterministic processor $g : y \mapsto z = g(y)$ such that,

$$
g : \mathcal{Y} \rightarrow \mathcal{Z} = \{0, 1\}
$$

$$
0 \mapsto 0
$$

$$
1 \mapsto 0
$$

$$
2 \mapsto 1
$$

$$
3 \mapsto 1.
$$

Clearly, $H(Z|X) = 0$ and $H(Z) = 1$. Therefore $I(X;Z) = 1$. We conclude that $I(X;Z) = I(X;Y)$ and $H(Z) < H(Y)$.

(b) Consider an error-free channel with binary input alphabet $\mathcal{X} = \{0, 1\}$ with $X$ uniformly distributed over $\mathcal{X}$, binary output alphabet $\mathcal{Y} = \{0, 1\}$, and probability law

$$
P_{Y|X}(y|x) = \begin{cases} 
1, & \text{if } x = y \\
0, & \text{otherwise}.
\end{cases}
$$

Choose now $\mathcal{Z} = \{0, 1, 2, 3\}$ and construct a probabilistic processor $G$ such that

$$
G : \mathcal{Y} \rightarrow \mathcal{Z}
$$

$$
0 \mapsto 0 \text{ with probability } \frac{1}{2} \text{ or } 1 \text{ with probability } \frac{1}{2}
$$

$$
1 \mapsto 2 \text{ with probability } \frac{1}{2} \text{ or } 3 \text{ with probability } \frac{1}{2}.
$$

Clearly, $I(X;Y) = 1 = I(X;Z)$ and $H(Y) = 1 < 2 = H(Z)$.

**Problem 4.**

(a)

$$
\Pr(U = u|V = ?) = \frac{\Pr(V = ?|U = u)p_{U}(u)}{\Pr(V = ?)} = \frac{p_{U}(u)p}{p} = p_{U}(u)
$$

(b)

$$
I(U;V) = H(U) - H(U|V)
$$

$$
= H(U) - \Pr(V = ?)H(U|V = ?) - \Pr(V \neq ?)H(U|V \neq ?)
$$

$$
\overset{(a)}{=} H(U) - p \sum_{u=1}^{K} \Pr(U = u|V = ?) \log \frac{1}{\Pr(U = u|V = ?)}
$$

$$
\overset{(b)}{=} H(U) - p \sum_{u=1}^{K} p_{U}(u) \log \frac{1}{p_{U}(u)} = H(U) - pH(U) = (1 - p)H(U),
$$

where (a) is obtained by noticing that if $V \neq ?$ then $V = U$ and $H(U|V \neq ?) = 0$ and (b) is obtained since $\Pr(U = u|V = ?) = p_{U}(u)$.
(c) Let $C_p$ be the capacity of this channel. Then,

$$C_p = \max_{pu} I(U, V) = \max_{pu} (1 - p) H(U) = (1 - p) \max_{pu} H(U) = (1 - p) \log K,$$

with the maximum achieved when $U$ is uniformly distributed over $\{1, \cdots, K\}$.

**Problem 5.**

(a) Since the channel is symmetric, the input distribution that maximizes the mutual information is the uniform one. Therefore, $C = 1 + \epsilon \log_2(\epsilon) + (1 - \epsilon) \log_2(\epsilon)$ which is equal to 0 when $\epsilon = \frac{1}{2}$.

(b) We have

- $I(X^n; Y^n) = I(X^n_2; Y^{n-1}) + I(X^n_2; Y_n|Y^{n-1}) + I(X_1; Y^n|X^n_2)$.
- $X^n_2 = Y^{n-1}$ and $Y_1, \ldots, Y_n$ are i.i.d. and uniform in $\{0, 1\}$, so $I(X^n_2; Y^{n-1}) = H(Y^{n-1}) = n - 1$.
- $Y_n$ is independent of $(X^n_2, Y^{n-1})$, so $I(X^n_2; Y_n|Y^{n-1}) = 0$.
- $X_1$ is independent of $(Y^n, X^n_2)$, so $I(X_1; Y^n|X^n_2) = 0$.

Therefore, $I(X^n; Y^n) = n - 1$.

(c) $W$ is independent of $Y^n$, so $I(W; Y^n) = 0 = nC$. 