ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

Handout 6 Information Theory and Coding
Homework 3 Oct. 02, 2023

PROBLEM 1. Recall that for a code $\mathcal{C} : \mathcal{U} \to \{0,1\}^*$, we define $\mathcal{C}^n : \mathcal{U}^n \to \{0,1\}^*$ as $\mathcal{C}^n(u_1 \dots u_n) = \mathcal{C}(u_1) \dots \mathcal{C}(u_n)$.

(a) Show that if $\mathcal{C}$ is uniquely decodable, then for all $n \geq 1$, $\mathcal{C}^n$ is injective.

(b) Suppose $\mathcal{C}$ is not uniquely decodable. Show that there are u^n and v^m such that $u_1 \neq v_1$ and $\mathcal{C}^n(u^n) = \mathcal{C}^m(v^m)$.

(c) Suppose $\mathcal{C}$ is not uniquely decodable. Show that there is a k such that $\mathcal{C}^k$ is not injective. [Hint: try $k = n + m$.]

PROBLEM 2. Suppose X,Y and Z are random variables.

(a) Show that $H(X) + H(Y) + H(Z) \geq \frac{1}{2}\big[H(XY) + H(YZ) + H(ZX)\big]$.

(b) Show that $H(XY) + H(YZ) \geq H(XYZ) + H(Y)$.

(c) Show that

$$2\big[H(XY) + H(YZ) + H(ZX)\big] \geq 3H(XYZ) + H(X) + H(Y) + H(Z).$$

(d) Show that $H(XY) + H(YZ) + H(ZX) \geq 2H(XYZ)$.

(e) Suppose n points in three dimensions are arranged so that their their projections to the xy, yz and zx planes give n_{xy}, n_{yz} and n_{zx} points. Clearly $n_{xy} \leq n$, $n_{yz} \leq n$, $n_{zx} \leq n$. Use part (d) show that

$$n_{xy} n_{yz} n_{zx} \geq n^2.$$

PROBLEM 3. Let X be a random variable taking values in M points $a_1, \dots, a_M$, and let $P_X(a_M) = \alpha$. Show that

$$H(X) = \alpha \log \frac{1}{\alpha} + (1-\alpha) \log \frac{1}{1-\alpha} + (1-\alpha) H(Y)$$

where Y is a random variable taking values in $M-1$ points $a_1, \dots, a_{M-1}$ with probabilities $P_Y(a_j) = P_X(a_j)/(1-\alpha)$; $1 \leq j \leq M-1$. Show that

$$H(X) \leq \alpha \log \frac{1}{\alpha} + (1-\alpha) \log \frac{1}{1-\alpha} + (1-\alpha) \log(M-1)$$

and determine the condition for equality.

PROBLEM 4. Let X, Y, Z be discrete random variables. Prove the validity of the following inequalities and find the conditions for equality:

(a) $I(X,Y;Z) \geq I(X;Z)$.

(b) $H(X,Y|Z) \geq H(X|Z)$.

(c) $H(X,Y,Z) - H(X,Y) \leq H(X,Z) - H(X)$.

(d) $I(X;Z|Y) \geq I(Z;Y|X) - I(Z;Y) + I(X;Z)$.

PROBLEM 5. For a stationary process $X_1, X_2, \ldots$, show that

(a) $\frac{1}{n} H(X_1, \ldots, X_n) \geq H(X_n | X_{n-1}, \ldots, X_1)$.

(b) $\frac{1}{n} H(X_1, \ldots, X_n) \leq \frac{1}{n-1} H(X_1, \ldots, X_{n-1})$.

PROBLEM 6. Let $\{X_i\}_{i=-\infty}^{\infty}$ be a stationary stochastic process. Prove that

$$H(X_0 | X_{-1}, \ldots, X_{-n}) = H(X_0 | X_1, \ldots, X_n).$$

That is: the conditional entropy of the present given the past is equal to the conditional entropy of the present given the future.

PROBLEM 7. Let $X \multimap Y \multimap (Z, W)$ form a Markov chain. Show that

$$I(X;Z) + I(X;W) \leq I(X;Y) + I(Z;W)$$