PROBLEM 1. Recall that for a code $C : \mathcal{U} \rightarrow \{0,1\}^*$, we define $C^n : \mathcal{U}^n \rightarrow \{0,1\}^*$ as $C^n(u_1 \ldots u_n) = C(u_1) \ldots C(u_n)$.

(a) Show that if $C$ is uniquely decodable, then for all $n \geq 1$, $C^n$ is injective.

(b) Suppose $C$ is not uniquely decodable. Show that there are $u^n$ and $v^m$ such that $u_1 \neq v_1$ and $C^n(u^n) = C^m(v^m)$.

(c) Suppose $C$ is not uniquely decodable. Show that there is a $k$ such that $C^k$ is not injective. [Hint: try $k = n + m$.]

PROBLEM 2. Suppose $X, Y$ and $Z$ are random variables.

(a) Show that $H(X) + H(Y) + H(Z) \geq \frac{1}{2} \left[ H(XY) + H(YZ) + H(XZ) \right]$.

(b) Show that $H(XY) + H(YZ) \geq H(XYZ) + H(Y)$.

(c) Show that $2[H(XY) + H(YZ) + H(XZ)] \geq 3H(XYZ) + H(X) + H(Y) + H(Z)$.

(d) Show that $H(XY) + H(YZ) + H(XZ) \geq 2H(XYZ)$.

(e) Suppose $n$ points in three dimensions are arranged so that their projections to the $xy$, $yz$ and $zx$ planes give $n_{xy}$, $n_{yz}$ and $n_{zx}$ points. Clearly $n_{xy} \leq n$, $n_{yz} \leq n$, $n_{zx} \leq n$. Use part (d) show that

$$n_{xy}n_{yz}n_{zx} \geq n^2.$$

PROBLEM 3. Let $X$ be a random variable taking values in $M$ points $a_1, \ldots, a_M$, and let $P_X(a_M) = \alpha$. Show that

$$H(X) = \alpha \log \frac{1}{\alpha} + (1 - \alpha) \log \frac{1}{1-\alpha} + (1 - \alpha)H(Y)$$

where $Y$ is a random variable taking values in $M-1$ points $a_1, \ldots, a_{M-1}$ with probabilities $P_Y(a_j) = P_X(a_j)/(1-\alpha); 1 \leq j \leq M-1$. Show that

$$H(X) \leq \alpha \log \frac{1}{\alpha} + (1 - \alpha) \log \frac{1}{1-\alpha} + (1 - \alpha)\log(M-1)$$

and determine the condition for equality.

PROBLEM 4. Let $X, Y, Z$ be discrete random variables. Prove the validity of the following inequalities and find the conditions for equality:

(a) $I(X, Y; Z) \geq I(X; Z)$.

(b) $H(X, Y|Z) \geq H(X|Z)$. 
(c) $H(X, Y, Z) - H(X, Y) \leq H(X, Z) - H(X)$.

(d) $I(X; Z|Y) \geq I(Z; Y|X) - I(Z; Y) + I(X; Z)$.

**Problem 5.** For a stationary process $X_1, X_2, \ldots$, show that

(a) $\frac{1}{n}H(X_1, \ldots, X_n) \geq H(X_n|X_{n-1}, \ldots, X_1)$.

(b) $\frac{1}{n}H(X_1, \ldots, X_n) \leq \frac{1}{n-1}H(X_1, \ldots, X_{n-1})$.

**Problem 6.** Let $\{X_i\}_{i=\infty}^{\infty}$ be a stationary stochastic process. Prove that

$$H(X_0|X_{-1}, \ldots, X_{-n}) = H(X_0|X_1, \ldots, X_n).$$

That is: the conditional entropy of the present given the past is equal to the conditional entropy of the present given the future.

**Problem 7.** Let $X \leftrightarrow Y \leftrightarrow (Z, W)$ form a Markov chain. Show that

$$I(X; Z) + I(X; W) \leq I(X; Y) + I(Z; W)$$