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Handout 22 Principles of Digital Communications
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PROBLEM 1. Suppose that we have two communication channels. On the first channel the transmitted vector $x \in \mathbb{R}^n$ is received as

$$Y = Ax + Z,$$

and on the second, the received vector is

$$Y = \tilde{A}x + Z,$$

where A and $\tilde{A}$ are deterministic $n \times n$ matrices and $Z \sim \mathcal{N}(0, \sigma^2 I_n)$.

Suppose that A and $\tilde{A}$ are related via $UA = \tilde{A}V$ where U and V are orthogonal matrices (i.e., $U^{-1} = U^T$ and similarly for V).

(a) Suppose we are given a transmitter (i.e., vectors $c_1, \dots, c_m$) and a receiver (i.e., a function $\hat{H} : \mathbb{R}^n \to \{1, \dots, m\}$ that gives the guessed message $\hat{H}(y)$ when the received vector is y) designed for the first channel. Show how we can design a transmitter and receiver for the second channel (i.e., $\{\tilde{c}_1, \dots, \tilde{c}_m\}$ and $\tilde{H}(y)$) such that

  (i) for every message i, $\|\tilde{c}_i\| = \|c_i\|$ (i.e., the new transmitter for the second channel is equivalent in energy to the first), and

  (ii) for every pair of messages i and k,

$$\Pr(\tilde{H}(Y) = k \mid \text{message } i \text{ is sent on the second channel}) = \Pr(\hat{H}(Y) = k \mid \text{message } i \text{ is sent on the first channel})$$

  (i.e., our design for the second channel has the same error probabilities as the given design for the first channel).

Justified by (a), we say that the two channels above are *equivalent*.

(b) Show that any channel of the form $Y = Ax + Z$ is equivalent to a channel $Y = \tilde{A}x + Z$ where $\tilde{A}$ is a diagonal matrix.

Consider now a channel where the input x and output Y are vectors of dimension nL, and Y is determined as follows:

1. The input $x = (x_1, \dots, x_{nL})$ is split into n segments $s_1 = (x_1, \dots, x_L)$, $s_2 = (x_{L+1}, \dots, x_{2L})$, $\dots$, $s_n = (x_{(n-1)L+1}, \dots, x_{nL})$, each of dimension L,

2. Each segment s_i is multiplied by an $L \times L$ matrix A,

3. These are then concatenated to form a vector of dimension nL, and

4. Y is formed by adding $Z \sim \mathcal{N}(0, \sigma^2 I_{nL})$ to the result of 3.

(An equivalent way to describe the above is to say that $Y = Bx + Z$ where $B = \text{diag}(A, \dots, A)$ is a block diagonal matrix.)

(c) Show that the channel above is equivalent to the channel $Y = \tilde{B}x + Z$ where

$$\tilde{B} = \operatorname{diag}(\underbrace{g_1, \dots, g_1}_{n}, \underbrace{g_2, \dots, g_2}_{n}, \dots, \underbrace{g_L, \dots, g_L}_{n})$$

for some $g_1, \dots, g_L$.

PROBLEM 2. Consider a channel where the transmitted vector $x \in \mathbb{R}^{2n}$ and received vector $Y \in \mathbb{R}^{2n}$ are related via

$$Y_i = g_i x_i + Z_i,$$

where $(Z_1, \dots, Z_{2n}) \sim \mathcal{N}(0, I_{2n})$. Suppose that $g_1 = g_2 = \cdots = g_n = 10$ and $g_{n+1} = g_{n+2} = \cdots = g_{2n} = 1$. So, the first half of the transmitted signal is amplified by a factor 10, while the second half experiences no amplification or attenuation.

We are asked to design a communication system with $m = 2^k$ messages, subject to the constraints that (i) our codewords $c_1, \dots, c_m$ satisfy

$$\frac{1}{m} \sum_{i=1}^{m} \|c_i\|^2 \le n,$$

and (ii) the error probability is less than some given ϵ. Our aim is to make k large subject to these constraints.

We adopt the following strategy: design two separate systems for each half of the transmission. That is, send $k = k_1 + k_2$ bits; where $k_1 = \log_2 m_1$ bits is sent during the first half of the transmission via vectors $c'_1, \dots, c'_{m_1}$, (all in $\mathbb{R}^n$) and $k_2 = \log_2 m_2$ bits is sent during the second half via vectors $c''_1, \dots, c''_{m_2}$ (again all in $\mathbb{R}^n$), so that the channel input is a concatenation of a c'_{i_1} and a c''_{i_2}. The receiver estimates i_1 from the first half $Y' = (Y_1, \dots, Y_n)$ of the received vector, and estimates i_2 from the second half $Y'' = (Y_{n+1}, \dots, Y_{2n})$.

(a) If $\mathcal{E}'$ and $\mathcal{E}''$ are the average energy of our first and second half designs, what is the average energy of the overall design?

(b) If P'_e and P''_e are the error probabilities for our designs for the first and second halves, what is the overall error probability?

For rest of the problem suppose $n = 100$ and that the error probability requirement is $\epsilon = 10^{-3}$.

(c) Consider a design with $k_2 = 0$ with $c''_1 = 0$ (i.e., the design only uses the first half). How large can we make $k = k_1$ with a QAM design? (In QAM, a message is sent using codewords in $\mathbb{R}^2$. Since $n = 100$ we can send 50 such messages — when choosing the QAM constellation don't forget that we need to ensure that the probability that all 50 messages are correctly received is high.)

(d) Consider a design with $\mathcal{E}' = \mathcal{E}''$. For the first half we use QAM, for the second we choose to use a repetition code. (A binary repetition code sends one bit via repeating $a \in \mathbb{R}$ or $-a$ r times, i.e., using codewords $(a, \dots, a)$ or $-(a, \dots, a)$. We can thus send $k_2 = \lfloor n/r \rfloor$ bits in the second half.) How large can we make $k = k_1 + k_2$?

(e) Suppose we make use of only the first two coordinates of the second half (in the remaining $n - 2$ coordinates we send 0). In these two coordinates we use a QAM constellation to send $k_2 = 2$ bits. For the first half we use QAM (as in parts (c) and (d)). How large can we make $k = k_1 + 2$?

(f) Repeat (e) with $k_2 = 4$.